UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Fall 2023

EXAM 1

Monday, September 25, 2023

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

NetID: ______________________________

Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Spectrum

$$\textbf{Scaling:}\ \ y(t)=Gx(t)=\sum_{k=-N}^{N}(Ga_k)\,e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\ \ y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\ \ \text{If } f_k=f'_n=f''_m \ \text{ then } \ a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\ \ y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k\tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\ \ y(t)=x(t)e^{j2\pi Ft}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\ \ y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}(j2\pi f_k a_k)\,e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases} z & f \bmod F_s<-f \bmod F_s \\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

1. (25 points) Suppose that

$$x(t) = 5\cos\left(3\pi t + \frac{\pi}{4}\right) + 5\cos\left(3\pi t - \frac{\pi}{4}\right) = \Re\left\{Me^{j(2\pi f t + \theta)}\right\},$$

where $\Re\{\cdot\}$ means "real part." Find M, f, and θ.

Solution:

$$\begin{aligned} x(t) &= 5\cos\left(3\pi t + \frac{\pi}{4}\right) + 5\cos\left(3\pi t - \frac{\pi}{4}\right) \\ &= \Re\left\{5e^{j(3\pi t + \frac{\pi}{4})} + 5e^{j(3\pi t - \frac{\pi}{4})}\right\} \\ &= \Re\left\{5(e^{j\pi/4} + e^{-j\pi/4})e^{j3\pi t}\right\} \\ &= \Re\left\{10\cos(\pi/4)e^{j3\pi t}\right\} \end{aligned}$$

So

$$\begin{aligned} f &= 1.5 \\ M &= 10\cos(\pi/4) = 5\sqrt{2} \\ \theta &= 0 \end{aligned}$$

2. (25 points) In a switching power supply, power is delivered to a circuit in the form of a periodic square wave, $x(t)$, with a period of T_0 seconds, and with an adjustable parameter R $(0 < R < 1)$ that is called the "duty cycle:"

$$x(t) = \begin{cases} 1 & 0 < t < RT_0 \\ 0 & RT_0 < t < T_0 \end{cases}$$

In terms of R, what are the Fourier series coefficients X_k? Calculate X_k for all values of k, including $k = 0$. There should be no unresolved integrals in your answer, and you should find that it is possible to express the answer without the variables t, T_0, or F_0; other simplifications are not necessary.

Solution: For $k = 0$, we have

$$\begin{aligned} X_0 &= \frac{1}{T_0}\int_0^{RT_0} dt \\ &= R \end{aligned}$$

For other values of k, we have

$$\begin{aligned} X_k &= \frac{1}{T_0}\int_0^{RT_0} e^{-j2\pi kF_0 t} dt \\ &= \frac{1}{-j2\pi kF_0T_0}\left[e^{-j2\pi kF_0 t}\right]_0^{RT_0} \\ &= \frac{1}{-j2\pi k}\left(e^{-j2\pi kF_0RT_0} - 1\right) \\ &= \frac{1}{-j2\pi k}\left(e^{-j2\pi kR} - 1\right) \end{aligned}$$

3. (25 points) The voltage across a cardioid microphone is

$$m(t) = ap(t) + (1-a)\frac{dp}{dt},$$

where $p(t)$ is the air pressure, as a function of time, at the microphone's location, and a is a coefficient, $0 < a < 1$, that depends on the direction from which the wave arrives. The microphone voltage is usually enhanced by a pre-amplifier with a gain of G, then transmitted on a wire until it reaches the A/D converter; the voltage at input to the A/D converter is delayed by a short delay of τ seconds, so

$$v(t) = Gm(t-\tau)$$

Suppose that the signal being recorded is the voice of a tenor, singing the A above middle C, thus

$$\begin{aligned}
p(t) &= \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t} \\
m(t) &= \sum_{k=-\infty}^{\infty} M_k e^{j2\pi k F_0 t} \\
v(t) &= \sum_{k=-\infty}^{\infty} V_k e^{j2\pi k F_0 t}
\end{aligned}$$

Express V_k in terms of X_k, F_0, k, a, G, τ, and k.

Solution:

$$\begin{aligned}
\frac{dp}{dt} &= \sum_{k=-\infty}^{\infty} j2\pi k F_0 X_k e^{j2\pi k F_0 t} \\
ap(t) + (1-a)\frac{dp}{dt} &= \sum_{k=-\infty}^{\infty} (a + (1-a)j2\pi k F_0) X_k e^{j2\pi k F_0 t} \\
Gm(t-\tau) &= \sum_{k=-\infty}^{\infty} G M_k e^{j2\pi k F_0 (t-\tau)} \\
&= \sum_{k=-\infty}^{\infty} G M_k e^{-j2\pi k F_0 \tau} e^{j2\pi k F_0 t} \\
&= \sum_{k=-\infty}^{\infty} G(a + (1-a)j2\pi k F_0) e^{-j2\pi k F_0 \tau} X_k e^{j2\pi k F_0 t}
\end{aligned}$$

Thus,

$$V_k = G(a + (1-a)j2\pi k F_0) e^{-j2\pi k F_0 \tau} X_k$$

4. (25 points) A signal $x(t)$ is sampled at 16,000 samples/second, then played back through an ideal D/A, thus:

$$
\begin{aligned}
x(t) &= 14\cos\left(2\pi 6000t + \frac{3\pi}{4}\right) + 2\cos\left(2\pi 12,000t - \frac{\pi}{4}\right)\\
x[n] &= x\left(t = \frac{n}{16,000}\right)\\
y(t) &= \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}\left(16,000\pi t - \pi n\right)
\end{aligned}
$$

where the sinc function is defined as

$$
\mathrm{sinc}(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & \text{otherwise} \end{cases}
$$

Find $y(t)$.

Solution:

$$
\begin{aligned}
x[n] &= 14\cos\left(\frac{2\pi 6000n}{16,000} + \frac{3\pi}{4}\right) + 2\cos\left(\frac{2\pi 12,000n}{16,000} - \frac{\pi}{4}\right)\\
&= 14\cos\left(\frac{12\pi n}{16} + \frac{3\pi}{4}\right) + 2\cos\left(\frac{24\pi n}{16} - \frac{\pi}{4}\right)\\
&= 14\cos\left(\frac{12\pi n}{16} + \frac{3\pi}{4}\right) + 2\cos\left(2\pi n - \left(\frac{24\pi n}{16} - \frac{\pi}{4}\right)\right)\\
&= 14\cos\left(\frac{12\pi n}{16} + \frac{3\pi}{4}\right) + 2\cos\left(\frac{8\pi n}{16} + \frac{\pi}{4}\right)\\
y(t) &= 14\cos\left(12,000\pi t + \frac{3\pi}{4}\right) + 2\cos\left(8000\pi t + \frac{\pi}{4}\right)
\end{aligned}
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