UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2022

EXAM 1

Wednesday, September 28, 2022

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ____

NetID: ____

Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Spectrum

$$\textbf{Scaling:}\ \ y(t)=Gx(t)=\sum_{k=-N}^{N}(Ga_k)\,e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\ \ y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\ \ \text{If } f_k=f'_n=f''_m\ \ \text{then}\ \ a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\ \ y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k\tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\ \ y(t)=x(t)e^{j2\pi Ft}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\ \ y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}(j2\pi f_k a_k)\,e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases}z & f \bmod F_s<-f \bmod F_s\\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

1. (25 points) Suppose that

$$x(t) = -12\cos\left(1000\pi t - \frac{\pi}{4}\right) + 4\sin\left(1000\pi t\right) = M\cos\left(1000\pi t + \theta\right)$$

Find x and y such that $M = \sqrt{x^2 + y^2}$ and either $\theta = \operatorname{atan}(y/x)$ or $\theta = \operatorname{atan}(y/x) - \pi$.

Solution:

$$x = -12\cos\left(-\frac{\pi}{4}\right) = -6\sqrt{2}$$
$$y = -12\sin\left(-\frac{\pi}{4}\right) - 4 = 6\sqrt{2} - 4$$

2. (25 points) $x(t)$ is a signal with a period of 0.01 seconds, and with the following shape:

$$x(t) = \begin{cases} 1 & 0 < t < 0.001 \\ 0 & 0.001 < t < 0.005 \\ -1 & 0.005 < t < 0.006 \\ 0 & 0.006 < t < 0.01 \end{cases}$$

(a) What are the Fourier series coefficients X_k for $k \neq 0$? Your answer should contain no variables other than k, but you don't need to simplify.

Solution:

$$\begin{aligned} X_k &= \frac{1}{-j2\pi k}\left(e^{-j\frac{2\pi k 0.001}{0.01}} - 1\right) - \frac{1}{-j2\pi k}\left(e^{-j\frac{2\pi k 0.006}{0.01}} - e^{-j\frac{2\pi k 0.005}{0.01}}\right) \\ &= \frac{j}{2\pi k}\left(e^{-j\frac{\pi k}{5}} - 1 - e^{-j\frac{6\pi k}{5}} + e^{-j\pi k}\right) \end{aligned}$$

...and this can be further simplified (a lot), but let's leave it here for now.

(b) Suppose that $y(t)$ is a signal such that $x(t) = \frac{dy}{dt}$. Express the Fourier series coefficients Y_k in terms of the Fourier series coefficients X_k. Note that you don't need to solve part (a) in order to solve this part of the problem.

Solution:

$$y(t) = \sum Y_k e^{j2\pi kt/T_0}$$
$$\frac{dy}{dt} = \sum \frac{j2\pi k}{0.01} Y_k e^{j2\pi kt/T_0}$$

...so therefore,

$$Y_k = \frac{0.01}{j2\pi k} X_k$$

3. (25 points) Suppose $x(t)$ is sampled to create the signal $y[n] = x\left(\frac{n}{F_s}\right)$, with a sampling frequency of $F_s = 10,000$ samples/second. The signal $y[n]$ is then passed through an ideal D/A in order to produce the signal $z(t)$. In each of the two following cases, what is $z(t)$?

(a)

$$x(t) = 3\cos\left(2\pi 8000t + \frac{\pi}{4}\right)$$

What is $z(t)$?

Solution: $f_a = F_s - f = 2000\text{Hz}$, so $z_a = z^* = 3e^{-j\pi/4}$, and

$$z(t) = 3\cos\left(2\pi 2000t - \frac{\pi}{4}\right)$$

(b)

$$x(t) = 3\cos\left(2\pi 12,000t + \frac{\pi}{4}\right)$$

What is $z(t)$?

Solution: $f_a = f - F_s = 2000$Hz, so $z_a = z = 3e^{j\pi/4}$, and

$$z(t) = 3\cos\left(2\pi 2000t + \frac{\pi}{4}\right)$$

4. (25 points) Suppose that

$$x[n] = \sin\left(\frac{\pi n}{2}\right) = \begin{cases} 1 & n \text{ is odd, and } \frac{n-1}{2} \text{ is even} \\ -1 & n \text{ is odd, and } \frac{n-1}{2} \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

You would like to generate a continuous-time audio signal, $y(t)$, using the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} x[n] g\left(t - n\right)$$

In each of the following cases, specify the value of $y(t)$ over the range $0 \le t \le 4$. You may specify $y(t)$ by drawing a plot of the function (if your plot clearly shows the value at each point in time in the range $0 \le t \le 4$), or by using an equation or a set of cases.

(a) What is $y(t)$ if

$$g(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$y(t) = \begin{cases} 1 & \frac{1}{2} \le t \le \frac{3}{2} \\ -1 & \frac{5}{2} \le t \le \frac{7}{2} \\ 0 & \text{otherwise} \end{cases}$$

(b) What is $y(t)$ if

$$g(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$y(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq 3 \\ t - 4 & 3 \leq t \leq 4 \end{cases}$$

(c) What is $y(t)$ if

$$g(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{otherwise} \end{cases}$$

Solution:

$$y(t) = \sin\left(\frac{\pi t}{2}\right)$$