UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2021

EXAM 3

Tuesday, December 14, 2021, 8:00-11:00am

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 200 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

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Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Fourier Series

$$\textbf{Analysis:}\;\; X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\;\; x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases} z & f \bmod F_s<-f \bmod F_s \\ z^* & f \bmod F_s>-f \bmod F_s \end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

Convolution

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

Frequency Response and DTFT

$$H(\omega)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(\omega)e^{j\omega n}d\omega$$

$$h[n]*\cos(\omega n)=|H(\omega)|\cos\left(\omega n+\angle H(\omega)\right)$$

Rectangular & Hamming Windows; Ideal LPF

$$w_R[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{otherwise}\end{cases}\leftrightarrow W_R(\omega)=e^{-\frac{j\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$w_H[n]=0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right)w_R[n]\leftrightarrow W_H(\omega)=0.54W_R(\omega)-0.23W_R\left(\omega-\frac{2\pi}{N-1}\right)-0.23W_R\left(\omega+\frac{2\pi}{N-1}\right)$$

$$h_{\text{ideal}}[n]=\frac{\omega_c}{\pi}\text{sinc}(\omega_c n)\leftrightarrow H_{\text{ideal}}(\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & \text{otherwise}\end{cases}$$

Discrete Fourier Transform

$$\textbf{Analysis:}\;\; X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}$$

$$\textbf{Synthesis:}\;\; x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N}$$

Z Transform Pairs

$$
\begin{aligned}
b_k z^{-k} &\leftrightarrow b_k \delta[n-k] \\
\frac{1}{1-az^{-1}} &\leftrightarrow a^n u[n] \\
\frac{1}{(1-e^{-\sigma_1 - \jmath\omega_1} z^{-1})(1-e^{-\sigma_1 + \jmath\omega_1} z^{-1})} &\leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1 (n+1)\right) u[n]
\end{aligned}
$$

1. (17 points) The variables A and θ are defined by the equation

$$4\cos\left(2600\pi t + \frac{\pi}{3}\right) + 3\sin\left(2600\pi t\right) = A\cos\left(2600\pi t + \theta\right)$$

Find explicit numerical expressions for A and θ.

2. (16 points) $x(t)$ and $y(t)$ are two signals that are both periodic with fundamental frequency F_0, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}, \qquad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j2\pi k F_0 t}$$

The relationship between $x(t)$ and $y(t)$ is given by

$$y(t) = x(t - 0.001) + 3\frac{dx}{dt}$$

Find Y_k in terms of X_k and F_0.

3. (17 points) A zebra is photographed at a resolution such that each black stripe is 3.5 pixels wide, and each white strip is 3.5 pixels wide. Modeling this as a square wave, we get the following model:

$$x[n] = 128 + 81\cos\left(\frac{\pi n}{7}\right) + 27\cos\left(\frac{3\pi n}{7}\right)$$

Unfortunately, the lens has some blur, so the image that is actually recorded is

$$y[n] = \frac{3}{4}x[n] + \frac{1}{4}x[n-1] \tag{1}$$

For some appropriate values of A_1, θ_1, A_3, and θ_3, $y[n]$ is equal to

$$y[n] = 128 + A_1\cos\left(\frac{\pi n}{7} + \theta_1\right) + A_3\cos\left(\frac{3\pi n}{7} + \theta_3\right) \tag{2}$$

Find explicit numerical expressions for A_1, θ_1, A_3, and θ_3 such that Eq. (2) is equal to Eq. (1).

4. (16 points) It's hard to convolve two sinc functions, but it's easy to muliply their Fourier transforms. For example, suppose that

$$x[n] = \text{sinc}(0.5\pi n)$$
$$h[n] = \frac{1}{3}\text{sinc}\left(\frac{\pi n}{3}\right)$$
$$y[n] = x[n] * h[n]$$

Find $y[n]$ as a function of n; x and h should not appear in your answer.

5. (17 points) Consider the signal

$$x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

Find $X[k]$, the 4-point DFT of $x[n]$. The variable k should not appear in your answer; instead, please write explicit numerical expressions for the four samples of $X[k]$, $k \in \{0, 1, 2, 3\}$.

6. (17 points) Suppose $x[n] = \cos(0.32\pi n) w_H[n]$, where $w_H[n]$ is a nearly-Hamming window defined as follows:

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{100}\right) & 0 \leq n \leq 99 \\ 0 & \text{otherwise} \end{cases}$$

Let $X[k]$ be the 100-point DFT of $x[n]$. For which values of k, $0 \leq k \leq 99$, is $X[k]$ nonzero?

7. (16 points) Suppose

$$x[n] = \begin{cases} 1 & 0 \le n \le 15 \text{ or } 49 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the 64-point DFT of $x[n]$, and

$$Y[k] = e^{-j\pi k} X[k]$$

Find $y[n]$, the inverse DTFT of $Y[k]$.

8. (17 points) Suppose $h[n]$ is a periodically repeated, rectangular-windowed ideal lowpass filter of length 94 samples

$$h[n] = \begin{cases} \frac{1}{3}\text{sinc}\left(\frac{\pi n}{3}\right) & 0 \le n \le 46 \\ \frac{1}{3}\text{sinc}\left(\frac{\pi}{3}(n-94)\right) & 47 \le n \le 93 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $x[n] = \delta[n-15]$, $X[k]$ is the DFT of $x[n]$, and $Y[k] = H[k]X[k]$. Find $y[n]$, the inverse DFT of $Y[k]$.

9. (17 points) Consider the difference equation

$$y[n] = x[n] - 0.6x[n-1] + 0.2x[n-2]$$

Find explicit numerical expressions for the frequencies, $\omega_1 = \angle z_1$ and $\omega_2 = \angle z_2$, of the zeros of this filter.

10. (16 points) Suppose you have designed a filter with the following system function:

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.7z^{-1}}$$

What is $h[n]$, the impulse response of this filter?

11. (17 points) A recording of a conversation, at $F_s = 16,000$ samples/second, is corrupted by an additive pure tone at exactly $f_1 = 1000$Hz. The tone is quite narrowband, so you can get rid of it using a notch filter with a 20Hz bandwidth. Write the difference equation for this notch filter; specify all coefficients as numbers, or as explicit numerical expressions.

12. (17 points) A particular signal has the following z-transform:

$$X(z) = \frac{1}{1 - 0.9e^{0.1\pi j}z^{-1}}$$

The signal, $x[n]$, is complex-valued in the time domain, with an exponentially decreasing absolute value. Find an explicit numerical expression for the smallest integer, n, such that

$$\frac{|x[n]|}{|x[0]|} \leq \frac{1}{e},$$

where e is the base of the natural logarithm.