UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2021

EXAM 1

Monday, September 27, 2021

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Spectrum

$$\textbf{Scaling:}\ \ y(t)=Gx(t)=\sum_{k=-N}^{N}\left(Ga_k\right)e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\ \ y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\ \ \text{If } f_k=f'_n=f''_m \ \ \text{then}\ \ a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\ \ y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k\tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\ \ y(t)=x(t)e^{j2\pi Ft}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\ \ y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}\left(j2\pi f_k a_k\right)e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s, -f \bmod F_s\right)$$

$$z_a=\begin{cases}z & f \bmod F_s < -f \bmod F_s\\ z^* & f \bmod F_s > -f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

1. (20 points) Suppose that

$$x(t) = 3\cos\left(2000\pi\left(t - \frac{1}{8000}\right)\right) + 4\sin\left(2000\pi t\right) = M\cos\left(2000\pi t + \theta\right)$$

Find x and y such that $M = \sqrt{x^2 + y^2}$ and $\theta = \text{atan}(y/x)$.

Solution:

$$\begin{aligned} x(t) &= 3\cos\left(2000\pi t - \frac{\pi}{4}\right) + 4\cos\left(2000\pi t - \frac{\pi}{2}\right) \\ &= \Re\left\{(3e^{-j\pi/4} + 4e^{-j\pi/2})e^{j2000\pi t}\right\} \end{aligned}$$

So

$$\begin{aligned} x &= 3\cos(-\pi/4) + 4\cos(-\pi/2) = \frac{3\sqrt{2}}{2} \\ y &= 3\sin(-\pi/4) + 4\sin(-\pi/2) = -\frac{3\sqrt{2}}{2} - 4 \end{aligned}$$

2. (20 points) $x(t)$ is a triangle wave with fundamental frequency of $F_0 = 100\text{Hz}$, and Fourier series coefficients of

$$X_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{j2\pi k^2} (-1)^{\frac{|k|-1}{2}} & k \text{ odd} \\ 0 & k \neq 0,\ k \text{ even} \end{cases}$$

The signal $y(t)$ is created by delaying $x(t)$ by 0.001 seconds, then differentiating it, i.e.,

$$y(t) = \frac{dx(t-0.001)}{dt}$$

What are the Fourier series coefficients Y_k?

Solution: Applying the spectral properties, we get

$$Y_k = \begin{cases} 0 & k = 0 \\ \frac{100}{k} (-1)^{\frac{|k|-1}{2}} e^{-j0.2\pi k} & k \text{ odd} \\ 0 & k \neq 0,\ k \text{ even} \end{cases}$$

3. (20 points) $x(t)$ is a signal with a period of 4 seconds, and with the following shape:

$$x(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \\ -2 & 2 < t < 3 \\ 0 & 3 < t < 4 \end{cases}$$

(a) What is F_0?

Solution:

$$F_0 = \frac{1}{4}\ \text{Hz}$$

(b) What is X_0, the 0^{th} Fourier coefficient?

Solution:

$$X_0 = \frac{1}{4}\int_0^4 x(t)dt = -\frac{1}{4}$$

(c) What are the Fourier series coefficients X_k for $k \neq 0$?

Solution:

$$\begin{aligned} X_k &= \frac{1}{4}\int_0^4 x(t)e^{-j2\pi kt/4}dt \\ &= \frac{1}{4}\left(\int_0^1 e^{-j2\pi kt/4}dt - 2\int_2^3 e^{-j2\pi kt/4}dt\right) \\ &= \frac{1}{4}\left(\frac{1}{-j2\pi k/4}\right)\left(\left[e^{-j2\pi kt/4}\right]_0^1 - 2\left[e^{-j2\pi kt/4}\right]_2^3\right) \\ &= \left(\frac{1}{-j2\pi k}\right)\left(e^{-j2\pi k/4} - 1 - 2e^{-j6\pi k/4} + 2e^{-j4\pi k/4}\right) \end{aligned}$$

4. (20 points) Suppose $x(t)$ is sampled to create the signal $y[n] = x\left(\frac{n}{F_s}\right)$, with a sampling frequency of $F_s = 8000$ samples/second. The signal $y[n]$ is then passed through an ideal D/A in order to produce the signal $z(t)$. In each of the two following cases, what is $z(t)$?

(a)

$$x(t) = 3\cos(2\pi 8000t)$$

Solution:

$$y[n] = 3\cos(2\pi 8000n/8000) = 3$$
$$z(t) = 3$$

(b)

$$x(t) = 2\cos\left(2\pi 4800t + \frac{\pi}{4}\right)$$

Solution:

$$\begin{aligned} y[n] &= 2\cos\left(\frac{2\pi 4800n}{8000} + \frac{\pi}{4}\right) \\ &= 2\cos\left(\frac{2\pi 3200n}{8000} - \frac{\pi}{4}\right) \\ z(t) &= 2\cos\left(2\pi 3200t - \frac{\pi}{4}\right) \end{aligned}$$

5. (20 points) Suppose that $x[n]$ is an audio signal sampled at $F_s = 10,000$ samples/second. You would like to generate a continuous-time audio signal, $y(t)$, using the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} x[n] g\left(t - \frac{n}{10000}\right)$$

You are considering several different levels of quality. In each of the following cases, write an equation for $g(t)$, or sketch it as a function of t. If you sketch $g(t)$, you must label its amplitude, and at least two of its zero-crossings.

(a) Suppose you want $y(t)$ to be a piece-wise constant interpolation of $x[n]$. What value of $g(t)$ would make this true?

Solution:

$$g(t) = \begin{cases} 1 & 0 \le t < \frac{1}{10000} \\ 0 & \text{otherwise} \end{cases}$$

(b) Suppose you want $y(t)$ to be a piece-wise linear interpolation of $x[n]$. What value of $g(t)$ would make this true?

Solution:

$$g(t) = \begin{cases} 1 - 10000|t| & -\frac{1}{10000} \le t \le \frac{1}{10000} \\ 0 & \text{otherwise} \end{cases}$$

(c) Suppose you want $y(t)$ to be a perfectly bandlimited reconstruction of of $x[n]$, with energy only in the frequency band $-5000 \le f \le 5000$. What value of $g(t)$ would make this true?

Solution:

$$g(t) = \frac{\sin(10000\pi t}{10000\pi t}$$