UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2021

EXAM 1

Monday, September 27, 2021

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

Phasors

$$A\cos(2\pi f t+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi f t}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi f t}+\frac{1}{2}e^{j\theta}e^{j2\pi f t}$$

Spectrum

$$\textbf{Scaling:}\ \ y(t)=Gx(t)=\sum_{k=-N}^{N}\left(Ga_k\right)e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\ \ y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\ \ \text{If } f_k=f'_n=f''_m \ \text{ then } \ a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\ \ y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k \tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\ \ y(t)=x(t)e^{j2\pi F t}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\ \ y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}\left(j2\pi f_k a_k\right)e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases}z & f \bmod F_s<-f \bmod F_s\\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

1. (20 points) Suppose that

$$x(t) = 3\cos\left(2000\pi\left(t - \frac{1}{8000}\right)\right) + 4\sin\left(2000\pi t\right) = M\cos\left(2000\pi t + \theta\right)$$

Find x and y such that $M = \sqrt{x^2 + y^2}$ and $\theta = \text{atan}(y/x)$.

2. (20 points) $x(t)$ is a triangle wave with fundamental frequency of $F_0 = 100$Hz, and Fourier series coefficients of

$$X_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{j2\pi k^2}(-1)^{\frac{|k|-1}{2}} & k \text{ odd} \\ 0 & k \neq 0,\ k \text{ even} \end{cases}$$

The signal $y(t)$ is created by delaying $x(t)$ by 0.001 seconds, then differentiating it, i.e.,

$$y(t) = \frac{dx(t-0.001)}{dt}$$

What are the Fourier series coefficients Y_k?

3. (20 points) $x(t)$ is a signal with a period of 4 seconds, and with the following shape:

$$x(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \\ -2 & 2 < t < 3 \\ 0 & 3 < t < 4 \end{cases}$$

(a) What is F_0?

(b) What is X_0, the 0^{th} Fourier coefficient?

(c) What are the Fourier series coefficients X_k for $k \neq 0$?

4. (20 points) Suppose $x(t)$ is sampled to create the signal $y[n] = x\left(\frac{n}{F_s}\right)$, with a sampling frequency of $F_s = 8000$ samples/second. The signal $y[n]$ is then passed through an ideal D/A in order to produce the signal $z(t)$. In each of the two following cases, what is $z(t)$?

(a)

$$x(t) = 3\cos(2\pi 8000 t)$$

(b)

$$x(t) = 2\cos\left(2\pi 4800 t + \frac{\pi}{4}\right)$$

5. (20 points) Suppose that $x[n]$ is an audio signal sampled at $F_s = 10,000$ samples/second. You would like to generate a continuous-time audio signal, $y(t)$, using the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} x[n] g\left(t - \frac{n}{10000}\right)$$

You are considering several different levels of quality. In each of the following cases, write an equation for $g(t)$, or sketch it as a function of t. If you sketch $g(t)$, you must label its amplitude, and at least two of its zero-crossings.

(a) Suppose you want $y(t)$ to be a piece-wise constant interpolation of $x[n]$. What value of $g(t)$ would make this true?

(b) Suppose you want $y(t)$ to be a piece-wise linear interpolation of $x[n]$. What value of $g(t)$ would make this true?

(c) Suppose you want $y(t)$ to be a perfectly bandlimited reconstruction of of $x[n]$, with energy only in the frequency band $-5000 \leq f \leq 5000$. What value of $g(t)$ would make this true?