UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 498MH Principles of Signal Analysis
Fall 2013

FINAL EXAM SOLUTIONS

Friday, December 13, 2013

- This is a CLOSED BOOK exam. You may use three pages (front and back) of your own notes, and you may use a calculator if you wish.
- There are a total of 200 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score	Problem	Score
1		6	
2		7	
3		8	
4		9	
5		10	
Total		Total	

Name: ____

Useful Angles

θ	$\cos\theta$	$\sin\theta$	$e^{j\theta}$
0	1	0	1
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/2 + j/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2 + j\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$	$1/2 + j\sqrt{3}/2$
$\pi/2$	0	1	j
π	-1	0	-1
$3\pi/2$	1	-1	$-j$
2π	1	0	1

Useful DTFTs

$$
\begin{aligned}
x[n] = a^n u[n] &\quad\leftrightarrow\quad X(\omega) = \frac{1}{1 - az^{-1}} \\
x[n] = \delta[n-k] &\quad\leftrightarrow\quad X(\omega) = e^{-j\omega k} \\
x[n] = e^{j\theta n} &\quad\leftrightarrow\quad X(\omega) = 2\pi\delta(\omega - \theta) \\
x[n] = \left(\frac{\omega_c}{\pi}\right)\operatorname{sinc}(\omega_c n) &\quad\leftrightarrow\quad X(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases} \\
x[n] = \begin{cases} 1 & |n| \le M \\ 0 & \text{otherwise} \end{cases} &\quad\leftrightarrow\quad X(\omega) = \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)}
\end{aligned}
$$

Problem 1 (20 points)

$$6\cos\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)+6\sin\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)=A\cos(\Omega t+\phi)$$

Find the following quantitites:

Solution:

$$A = \boxed{6\sqrt{2}}$$

$$\Omega = \boxed{2\pi 1000}$$

$$\phi = \boxed{-\frac{3\pi}{4}}$$

Problem 2 (20 points)

A periodic signal $x(t)$, with period T_0, is given by

$$x(t) = \begin{cases} 1 & 0 \leq t \leq \frac{3T_0}{4} \\ 0 & \frac{3T_0}{4} < t < T_0 \end{cases}$$

The same signal can be expressed as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Find $|X_2|$, the amplitude of the second harmonic.

Solution:

$$|X_2| = \frac{1}{2\pi}$$

Problem 3 (20 points)

A particular system generates an output $y[n]$ from its input $x[n]$ according to the following rule:

$$y[n] = \begin{cases} x[n] & n \text{ is even} \\ \frac{1}{2}\left(x[n-1] + x[n+1]\right) & n \text{ is odd} \end{cases}$$

(a) (**6 points**) Is the system linear? Give your reason.

(b) (**4 points**) Is the system causal? Give your reason.

(c) (**6 points**) Is the system time-invariant? Give your reason.

(d) (**4 points**) Is the system stable? Give your reason.

Problem 4 (20 points)

Find $y[n] = h[n] * x[n]$, where

$$x[n] = \cos(0.02\pi n)\,, \quad h[n] = \begin{cases} 1 & |n| \le 3 \\ 0 & |n| > 3 \end{cases}$$

What is $y[n]$? Hint: Find $H(\omega)$ first. In order to find the numerical value of your answer, you may find it useful to approximate $\sin x \approx x$, an approximation that works for small values of x.

Problem 5 (20 points)

Find $y[n] = h[n] * x[n]$, where

$$x[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}, \quad h[n] = \begin{cases} 1 & |n| \leq 3 \\ 0 & |n| > 3 \end{cases}$$

What is $y[n]$?

Problem 6 (20 points)

Suppose

$$x[n] = \cos\left(\frac{7\pi n}{21}\right), \quad y[n] = \begin{cases} x[n] & |n| \le 10 \\ 0 & \text{otherwise} \end{cases}, \quad Y(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

Sketch $Y(\omega)$ for $-\pi \le \omega \le \pi$. Specify the frequency and amplitude of at least one peak. Also, specify at least three particular frequencies ω such that $Y(\omega) = 0$.

Problem 7 (20 points)

You have a 250×250 image that you want to upsample to 250×1000 without introducing any aliasing. If $x[n]$ is a row of the original image, and $y[n]$ is a row of the upsampled image, this task can be accomplished by

$$y[n] = \sum_{m=0}^{249} x[m]g[n-4m]$$

Sketch $g[n]$ as a function of n. Show the value of $g[0]$, and specify at least three particular sample indices, n, at which $g[n] = 0$.

Problem 8 (20 points)

An 8000Hz tone, $x(t) = \cos(2\pi 8000t)$, is sampled at $F_s = \frac{1}{T} = 10,000$ samples/second in order to create $x[n] = x(nT)$. Sketch $X(\omega)$ for $0 \leq \omega \leq 2\pi$ (**note the domain!!**). Specify the frequencies at which $X(\omega) \neq 0$.

Solution:

$X(\omega)$ is a spectrum with energy at the frequencies $(0.4\pi, 1.6\pi)$.

Problem 9 (20 points)

Suppose $x[n]$ is a random signal with the following autocorrelation:

$$R_{xx}[\tau] = \frac{1}{16}\text{sinc}^2\left(\frac{\pi n}{4}\right) = \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2$$

Suppose $e[n] = x[n] - ax[n-1]$, and you want to find a in order to minimize $E\left[e^2[n]\right]$. Find the numerical value of a ("numerical" in the sense that there are no variables in your answer, however, your answer may include constants like π and $\sqrt{2}$).

Problem 10 (20 points)

Suppose $y[n] = x[n] + v[n]$. $v[n]$ is zero-mean, unit-variance white noise uncorrelated with $x[n]$, and $x[n]$ is a random signal whose power spectrum is given by

$$P_{xx}(\omega) = \begin{cases} \frac{\pi}{2} - |\omega| & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

Suppose $z[n] = h[n] * y[n]$. Find $H(\omega)$ in order to minimize $E\left[(z[n] - x[n])^2\right]$.