

Lecture 28: IIR Filters

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ECE 401: Signal and Image Analysis

- 1 Review: Z Transform
- 2 Autoregressive Difference Equations
- 3 Finite vs. Infinite Impulse Response
- 4 Impulse Response and Transfer Function of a First-Order Autoregressive Filter
- 5 Finding the Poles and Zeros of $H(z)$
- 6 Summary

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Summary: Z Transform

- A **difference equation** is an equation in terms of time-shifted copies of $x[n]$ and/or $y[n]$.
- We can find the frequency response $H(\omega) = Y(\omega)/X(\omega)$ by taking the DTFT of each term of the difference equation. This will result in a lot of terms of the form $e^{j\omega n_0}$ for various n_0 .
- We have less to write if we use a new frequency variable, $z = e^{j\omega}$. This leads us to the Z transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Zeros of the Transfer Function

- The **transfer function**, $H(z)$, is a polynomial in z .
- The zeros of the transfer function are usually complex numbers, z_k .
- The frequency response, $H(\omega) = H(z)|_{z=e^{j\omega}}$, has a dip whenever ω equals the phase of any of the zeros, $\omega = \angle z_k$.

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Autoregressive Difference Equations

An **autoregressive** filter is one in which the output, $y[n]$, depends on past values of itself (**auto**=self, **regress**=go back). For example,

$$y[n] = x[n] + 0.3x[n - 1] + 0.8y[n - 1]$$

Causal and Anti-Causal Filters

- If the outputs of a filter depend only on **current and past** values of the input, then the filter is said to be **causal**. An example is

$$y[n] = x[n] + 0.3x[n - 1] + 0.8y[n - 1]$$

- If the outputs depend only on **current and future** values of the input, the filter is said to be **anti-causal**, for example

$$y[n] = x[n] + 0.3x[n + 1] + 0.8y[n + 1]$$

- If the filter is neither causal nor anti-causal, we say it's "non-causal."
- Feedforward non-causal filters are easy to analyze, but when analyzing feedback, we will stick to causal filters.

Autoregressive Difference Equations

We can find the transfer function by taking the Z transform of each term in the equation:

$$y[n] = x[n] + 0.3x[n - 1] + 0.8y[n - 1]$$

$$Y(z) = X(z) + 0.3z^{-1}X(z) + 0.8z^{-1}Y(z)$$

Transfer Function

In order to find the transfer function, we need to solve for

$$H(z) = \frac{Y(z)}{X(z)}.$$

$$Y(z) = X(z) + 0.3z^{-1}X(z) + 0.8z^{-1}Y(z)$$

$$(1 - 0.8z^{-1}) Y(z) = X(z)(1 + 0.3z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.3z^{-1}}{1 - 0.8z^{-1}}$$

Frequency Response

As before, we can get the frequency response by just plugging in $z = e^{j\omega}$. Some autoregressive filters are unstable,¹ but if the filter is stable, then this works:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \frac{1 + 0.3e^{-j\omega}}{1 - 0.8e^{-j\omega}}$$

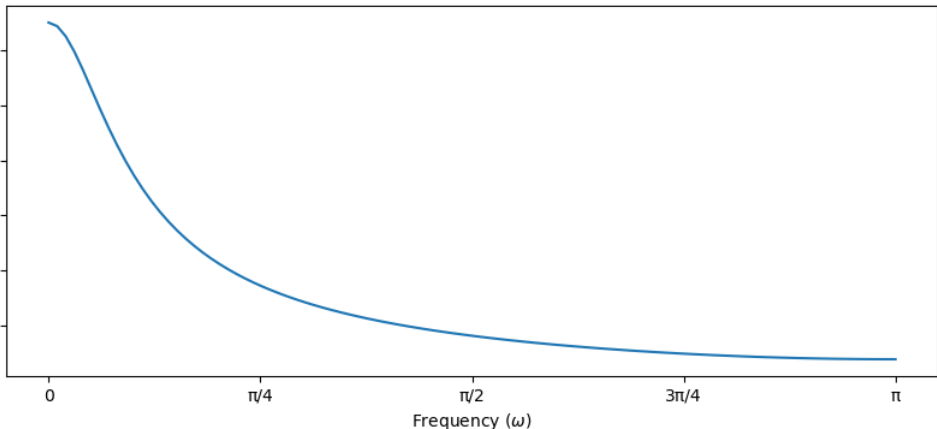
¹“Unstable” means that the output can be infinite, even with a finite input. More about this later in the lecture.

Frequency Response

So, already we know how to compute the frequency response of an autoregressive filter. Here it is, plotted using

```
np.abs((1+0.3*np.exp(-1j*omega))/(1-0.8*np.exp(-1j*omega)))
```

$$|H(\omega)| = (1 + 0.3e^{j\omega}) / (1 - 0.8e^{-j\omega})$$



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Impulse Response of an Autoregressive Filter

One way to find the **impulse response** of an autoregressive filter is the same as for any other filter: feed in an impulse, $x[n] = \delta[n]$, and what comes out is the impulse response, $y[n] = h[n]$.

$$h[n] = \delta[n] + 0.3\delta[n-1] + 0.8h[n-1]$$

$$h[n] = 0, \quad n < 0$$

$$h[0] = \delta[0] = 1$$

$$h[1] = 0 + 0.3\delta[0] + 0.8h[0] = 1.1$$

$$h[2] = 0 + 0 + 0.8h[1] = 0.88$$

$$h[3] = 0 + 0 + 0.8h[2] = 0.704$$

$$\vdots$$

$$h[n] = 1.1(0.8)^{n-1} \quad \text{if } n \geq 1$$

$$\vdots$$

FIR vs. IIR Filters

- An autoregressive filter is also known as an **infinite impulse response (IIR)** filter, because $h[n]$ is infinitely long (never ends).
- A difference equation with only feedforward terms (like we saw in the last lecture) is called a **finite impulse response (FIR)** filter, because $h[n]$ has finite length.

General form of an FIR filter

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

This filter has an impulse response ($h[n]$) that is $M + 1$ samples long.

- The b_k 's are called **feedforward** coefficients, because they feed $x[n]$ forward into $y[n]$.

General form of an IIR filter

$$\sum_{\ell=0}^N a_{\ell} y[n-\ell] = \sum_{k=0}^M b_k x[n-k]$$

- The a_{ℓ} 's are called **feedback** coefficients, because they feed $y[n]$ back into itself.

General form of an IIR filter

$$\sum_{\ell=0}^N a_{\ell} y[n - \ell] = \sum_{k=0}^M b_k x[n - k]$$

Example:

$$y[n] = x[n] + 0.3x[n - 1] + 0.8y[n - 1]$$

$$b_0 = 1$$

$$b_1 = 0.3$$

$$a_0 = 1$$

$$a_1 = -0.8$$

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First-Order Feedback-Only Filter

Let's find the general form of $h[n]$, for the simplest possible autoregressive filter: a filter with one feedback term, and no feedforward terms, like this:

$$y[n] = x[n] + ay[n - 1],$$

where a is any constant (positive, negative, real, or complex).

Impulse Response of a First-Order Filter

We can find the impulse response by putting in $x[n] = \delta[n]$, and getting out $y[n] = h[n]$:

$$h[n] = \delta[n] + ah[n - 1].$$

Recursive computation gives

$$h[0] = 1$$

$$h[1] = a$$

$$h[2] = a^2$$

$$\vdots$$

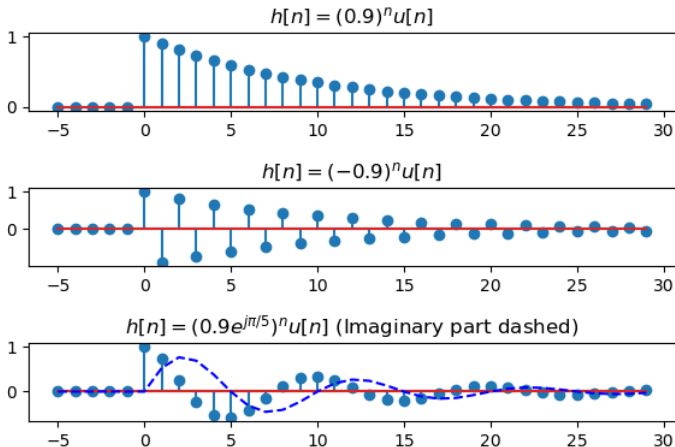
$$h[n] = a^n u[n]$$

where we use the notation $u[n]$ to mean the “unit step function,”

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

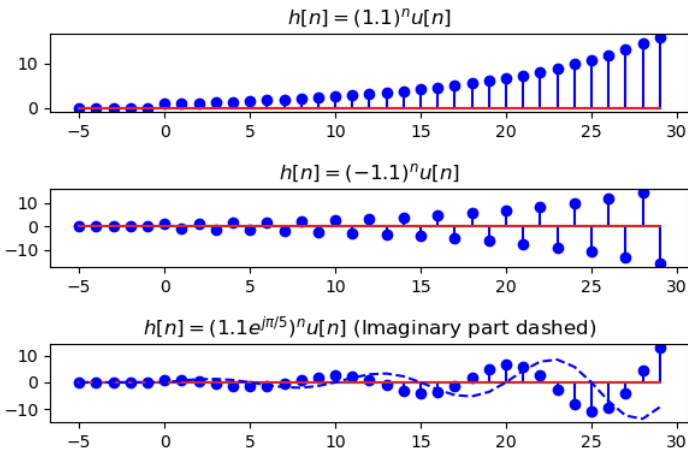
Impulse Response of Stable First-Order Filters

The coefficient, a , can be positive, negative, or even complex. If a is complex, then $h[n]$ is also complex-valued.



Impulse Response of Unstable First-Order Filters

If $|a| > 1$, then the impulse response grows exponentially. If $|a| = 1$, then the impulse response never dies away. In either case, we say the filter is “unstable.”



Instability

- A **stable** filter is one that always generates finite outputs ($|y[n]|$ finite) for every possible finite input ($|x[n]|$ finite).
- An **unstable** filter is one that, at least sometimes, generates infinite outputs, even if the input is finite.
- A first-order IIR filter is stable if and only if $|a| < 1$.

Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation equation:

$$y[n] = x[n] + ay[n - 1],$$
$$Y(z) = X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

Frequency Response of a First-Order Filter

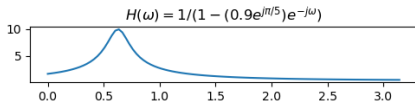
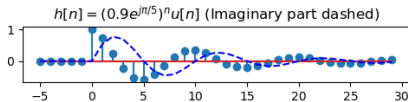
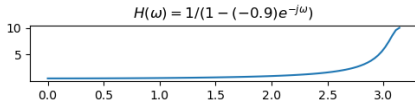
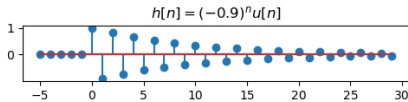
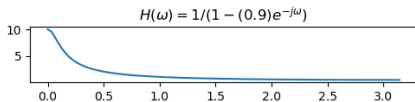
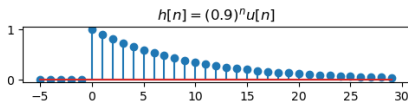
If the filter is stable ($|a| < 1$), then we can find the frequency response by plugging in $z = e^{j\omega}$:

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \frac{1}{1 - ae^{-j\omega}} \quad \text{iff } |a| < 1$$

This formula works if and only if $|a| < 1$.

Frequency Response of a First-Order Filter

$$H(\omega) = \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |a| < 1$$



Transfer Function \leftrightarrow Impulse Response

For FIR filters, we say that $h[n] \leftrightarrow H(z)$ are a Z-transform pair. Let's assume that the same thing is true for IIR filters, and see if it works.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \end{aligned}$$

This is a standard geometric series, with a ratio of az^{-1} . As long as $|a| < 1$, we can use the formula for an infinite-length geometric series, which is:

$$H(z) = \frac{1}{1 - az^{-1}},$$

So we confirm that $h[n] \leftrightarrow H(z)$ for both FIR and IIR filters, as long as $|a| < 1$.

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First-Order Filter

Now, let's find the transfer function of a general first-order filter, including BOTH feedforward and feedback delays:

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$

where we'll assume that $|a| < 1$, so the filter is stable.

Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation:

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$
$$Y(z) = X(z) + bz^{-1}X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + bz^{-1}}{1 - az^{-1}}.$$

Treating $H(z)$ as a Ratio of Two Polynomials

Notice that $H(z)$ is the ratio of two polynomials:

$$H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \frac{z + b}{z - a}$$

- $z = -b$ is called the **zero** of $H(z)$, meaning that $H(-b) = 0$.
- $z = a$ is called the **pole** of $H(z)$, meaning that $H(a) = \infty$

The Pole and Zero of $H(z)$

- The pole, $z = a$, and zero, $z = -b$, are the values of z for which $H(z) = \infty$ and $H(z) = 0$, respectively.
- But what does that mean? We know that for $z = e^{j\omega}$, $H(z)$ is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the pole and zero do not normally have unit magnitude.

- What it means is that:
 - When $\omega = \angle(-b)$, then $|H(\omega)|$ is as close to a zero as it can possibly get, so at that that frequency, $|H(\omega)|$ is as low as it can get.
 - When $\omega = \angle a$, then $|H(\omega)|$ is as close to a pole as it can possibly get, so at that that frequency, $|H(\omega)|$ is as high as it can get.

Review

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Autoregressive

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FIR and IIR

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First-Order

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Poles and Zeros

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Poles and Zeros

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Vectors in the Complex Plane

Suppose we write $|H(z)|$ like this:

$$|H(z)| = \frac{|z + b|}{|z - a|}$$

Now let's evaluate at $z = e^{j\omega}$:

$$|H(\omega)| = \frac{|e^{j\omega} + b|}{|e^{j\omega} - a|}$$

What we've discovered is that $|H(\omega)|$ is small when the vector distance $|e^{j\omega} + b|$ is small, but LARGE when the vector distance $|e^{j\omega} - a|$ is small.

Review
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Autoregressive
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FIR and IIR
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First-Order
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Poles and Zeros
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Why This is Useful

Now we have another way of thinking about frequency response.

- Instead of just LPF, HPF, or BPF, we can design a filter to have zeros at particular frequencies, $\angle(-b)$, AND to have poles at particular frequencies, $\angle a$,
- The magnitude $|H(\omega)|$ is $|e^{j\omega} + b|/|e^{j\omega} - a|$.
- Using this trick, we can design filters that have much more subtle frequency responses than just an ideal LPF, BPF, or HPF.

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Summary: Autoregressive Filter

- An **autoregressive filter** is a filter whose current output, $y[n]$, depends on past values of the output.
- An autoregressive filter is also called **infinite impulse response (IIR)**, because $h[n]$ has infinite length.
- A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).
- The first-order, feedback-only autoregressive filter has this impulse response and transfer function:

$$h[n] = a^n u[n] \leftrightarrow H(z) = \frac{1}{1 - az^{-1}}$$

Summary: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$

has the impulse response and transfer function

$$h[n] = a^n u[n] + ba^{n-1} u[n - 1] \leftrightarrow H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}},$$

where a is called the **pole** of the filter, and $-b$ is called its **zero**.