Lecture 25: Exam 2 Review

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ECE 401: Signal and Image Analysis

1. Topics Covered
2. Convolution
3. Frequency Response
4. Discrete Time Fourier Transform
5. Summary

Outline

1. Topics Covered
2. Convolution
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Topics Covered

- Convolution
  - Convolution, Impulse Response
  - Linearity, Shift-Invariance, Causality, Stability
- DTFT
  - Frequency Response, DTFT, Cascaded Systems
  - Ideal Filters, Windowing, Rectangular Window, Hamming Window

Not included: Bartlett & Hann windows; spectral analysis

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Linearity and Shift-Invariance

- A system is **linear** if and only if, for any two inputs $x_1[n]$ and $x_2[n]$ that produce outputs $y_1[n]$ and $y_2[n]$,

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

- A system is **shift-invariant** if and only if, for any input $x_1[n]$ that produces output $y_1[n]$,

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$

- If a system is **linear and shift-invariant** (LSI), then it can be implemented using convolution:

$$y[n] = h[n] * x[n] = \sum_m h[m]x[n-m] = \sum_m h[n-m]x[m]$$

where $h[n]$ is the impulse response:

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$

Causality and Stability

- A system is causal if and only if $h[n]$ is right-sided.
  - A causal system has a negative phase response.
- A system is stable if and only if $h[n]$ is magnitude-summable.
  - A stable system has a finite magnitude response.

Finite Impulse Response

- A **finite impulse response (FIR)** filter is one whose impulse response has finite length.
- If $h[n]$ has finite length, then we can implement the filter using an explicit summation:

$$y[n] = h[n] * x[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

Example: First Difference

The first difference operator is:

$$x[n] \xrightarrow{\mathcal{H}} y[n] = x[n] - x[n-1]$$

Its impulse response is:

$$h[n] = \delta[n] - \delta[n-1]$$

Example: Simple Delay

The delay operator is:

$$x[n] \xrightarrow{\mathcal{H}} y[n] = x[n - n_0]$$

Its impulse response is:

$$h[n] = \delta[n - n_0]$$

Example: Local Adder

The local adder is:

$$x[n] \xrightarrow{\mathcal{H}} y[n] = \sum_{m=0}^{N-1} x[n-m]$$

Its impulse response is:

$$h[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

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Frequency Response

- **Tones in $\rightarrow$ Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)|\cos(\omega n + \angle H(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)|\cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Example: First Difference

The first difference impulse response is:

$$h[n] = \delta[n] - \delta[n-1]$$

Its frequency response is:

$$\begin{aligned} H(\omega) &= 1 - e^{-j\omega} \\ &= e^{j\left(\frac{\pi-\omega}{2}\right)} 2\sin(\omega/2) \end{aligned}$$

Example: Simple Delay

The delay operator is:

$$h[n] = \delta[n - n_0]$$

Its frequency response is:

$$H(\omega) = e^{-j\omega n_0}$$

Example: Local Adder

The delayed local adder's impulse response is:

$$h[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Its frequency response is:

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\ &= e^{-\frac{j\omega(N-1)}{2}} D_N(\omega), \end{aligned}$$

where $D_N(\omega)$ is the "Dirichlet form," sometimes called the "digital sinc:"

$$D_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Response of an LSI System to a Periodic Input

If the input of an LSI system is periodic,

$$x[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

…then the output is

$$y[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k H(k\omega_0) e^{j2\pi kn/N_0}$$

Cascaded LSI Systems

Cascaded LSI Systems convolve their impulse responses, equivalently, they multiply their frequency responses:

$$y[n] = h[n] * g[n] * x[n] \quad \leftrightarrow \quad Y_k = H(k\omega_0)G(k\omega_0)X_k$$

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The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n-n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X(\omega + 2\pi) = X(\omega)$
1. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

2. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
3. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
4. Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Ideal Filters

- Ideal Lowpass Filter:

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c, \\ 0 & \omega_c < |\omega| \le \pi. \end{cases} \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Bandpass Filter:

$$H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega)$$
$$\leftrightarrow h_{BP}[n] = \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n)$$

- Ideal Highpass Filter:

$$H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \text{sinc}(\pi n) - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

Practical Filters

- Even-symmetric in time (odd length only):

$$h[n] = \begin{cases} h_{\text{ideal}}[n]w[n] & -\frac{N-1}{2} \le n \le \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Right-sided in time (odd or even length):

$$h[n] = \begin{cases} h_{\text{ideal}}\left[n - \left(\frac{N-1}{2}\right)\right] w[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

where $w[n]$ is a finite-length windowing function.

Windows

You need to know these two windows, presented here in their right-sided forms:

- Rectangular Window:

$$w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad W(\omega) = e^{-\frac{j\omega(N-1)}{2}} D_N(\omega)$$

  - Main lobe halfwidth (first null): $\frac{2\pi}{N}$, therefore transition bandwidths are $\frac{4\pi}{N}$
  - First sidelobe level: -13dB, therefore stopband ripple is <-13dB

- Hamming Window:

$$w[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

  - Main lobe halfwidth (first null): $\frac{4\pi}{N}$, therefore transition bandwidths are $\frac{8\pi}{N}$
  - First sidelobe level: -44dB, therefore stopband ripple is <-44dB

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Summary: Topics Covered

DSP First, chapters 5-7:

1. Chapter 5: FIR Filters
   - LSI systems, impulse response, convolution
   - first difference, pure delay, local sum
2. Chapter 6: Frequency Response
   - complex exponentials, cosines, periodic signals
   - cascaded systems
   - first difference, pure delay, local sum
3. Chapter 7: DTFT
   - DTFT & frequency response
   - Ideal filters
   - Practical filters; rectangular & Hamming windows