Lecture 20: Windows

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ECE 401: Signal and Image Analysis, Fall 2023

Outline

How to create a realizable digital filter

- $L =$ Odd Length:
$$h[n] = h_i[n]w[n],$$
where $w[n]$ is nonzero for $-\left(\frac{L-1}{2}\right) \leq n \leq \left(\frac{L-1}{2}\right)$
- $L =$ Even Length:
$$h[n] = h_i\left[n - \left(\frac{L-1}{2}\right)\right] w[n]$$
where $w[n]$ is nonzero for $0 \leq n \leq L-1$.

Multiplication $\leftrightarrow$ Convolution!

- Convolution $\leftrightarrow$ Multiplication:

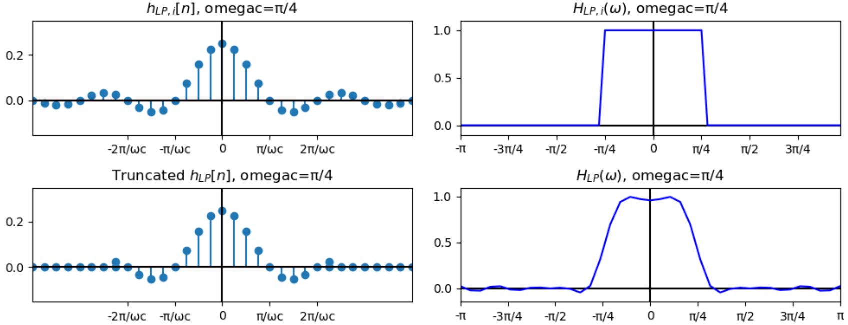
$$h[n] * x[n] \leftrightarrow H(\omega)X(\omega)$$

- Multiplication $\leftrightarrow$ Convolution:

$$w[n]h[n] \leftrightarrow \frac{1}{2\pi} W(\omega) * H(\omega)$$

Result: Windowing Causes Artifacts

We've already seen the result. Windowing by a rectangular window (i.e., truncation) causes nasty artifacts!



Windowing Causes Artifacts

$$h[n] = h_i[n]w[n] \leftrightarrow H(\omega) = \frac{1}{2\pi} H_i(\omega) * W(\omega)$$

Today's Topic:

What is $W(\omega)$?

Outline

Review: Rectangle $\leftrightarrow$ Sinc

- The DTFT of a sinc is a rectangle:

$$h[n] = \left(\frac{\omega_c}{\pi}\right)\mathrm{sinc}(\omega_c n) \quad \leftrightarrow \quad H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

- The DTFT of a rectangle is a sinc-like function, called the Dirichlet form:

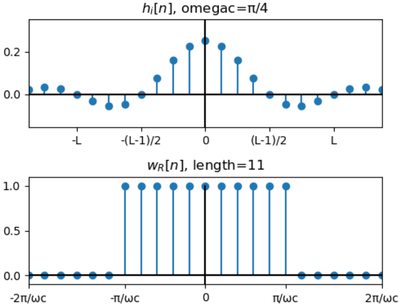
$$w_R[n] = \begin{cases} 1 & |n| \le \frac{L-1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Dirichlet Form: Proof Review

Review of the proof:

$$
\begin{aligned}
W_R(\omega) &= \sum_{n=-\infty}^{\infty} w_R[n]e^{-j\omega n} = \sum_{n=-\frac{L-1}{2}}^{\frac{L-1}{2}} e^{-j\omega n} \\
&= e^{j\omega\left(\frac{L-1}{2}\right)} \sum_{m=0}^{L-1} e^{-j\omega m} \\
&= e^{j\omega\left(\frac{L-1}{2}\right)} \left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right) \\
&= \left(\frac{e^{j\omega L/2}-e^{-j\omega L/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\
&= \left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right)
\end{aligned}
$$

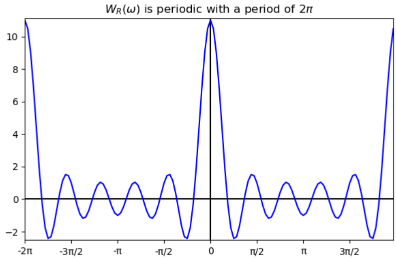
Review: Rectangle ↔ Sinc



Properties of the Dirichlet form: Periodicity

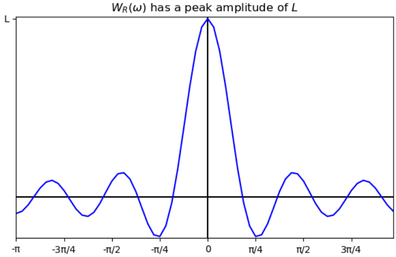
$$W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Both numerator and denominator are periodic with period 2π.



Properties of the Dirichlet form: DC Value

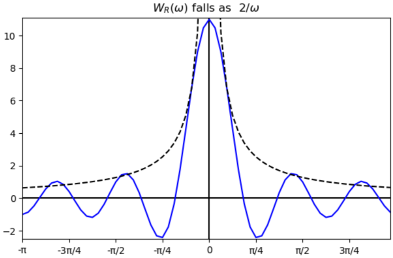
$$W_R(0) = \sum_{n=-\infty}^{\infty} w[n] = L$$



Properties of the Dirichlet form: Sinc-like

$$W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$
$$\approx \frac{\sin(\omega L/2)}{\omega/2}$$

Because, for small values of ω, $\sin\left(\frac{\omega}{2}\right) \approx \frac{\omega}{2}$.



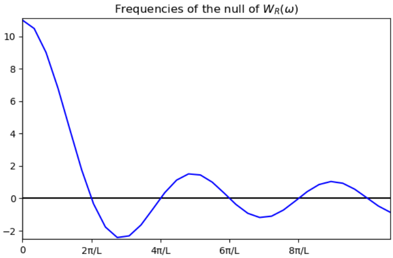
Properties of the Dirichlet form: Nulls

$$W_R(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

It equals zero whenever

$$\frac{\omega L}{2} = k\pi$$

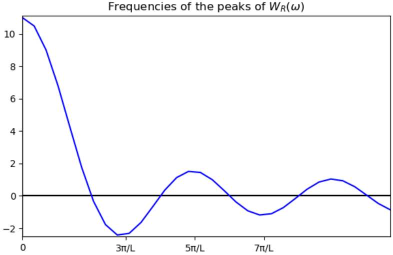
For any nonzero integer, k.



Properties of the Dirichlet form: Sidelobes

Its sidelobes are

$$W_R\left(\frac{3\pi}{L}\right) = \frac{-1}{\sin(3\pi/2L)} \approx \frac{-2L}{3\pi}$$
$$W_R\left(\frac{5\pi}{L}\right) = \frac{1}{\sin(5\pi/2L)} \approx \frac{2L}{5\pi}$$
$$W_R\left(\frac{7\pi}{L}\right) = \frac{-1}{\sin(7\pi/2L)} \approx \frac{-2L}{7\pi}$$
$$\vdots$$



Properties of the Dirichlet form: Relative Sidelobe Amplitudes

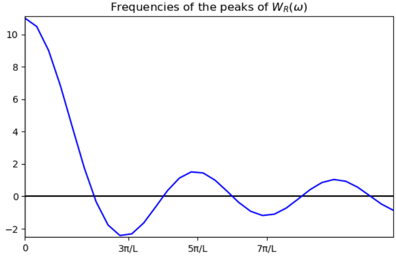
The **relative** sidelobe amplitudes don't depend on L:

$$\frac{W_R\left(\frac{3\pi}{L}\right)}{W_R(0)} = \frac{-1}{L\sin(3\pi/2L)} \approx \frac{-2}{3\pi}$$

$$\frac{W_R\left(\frac{5\pi}{L}\right)}{W_R(0)} = \frac{1}{L\sin(5\pi/2L)} \approx \frac{2}{5\pi}$$

$$\frac{W_R\left(\frac{7\pi}{L}\right)}{W_R(0)} = \frac{-1}{L\sin(7\pi/2L)} \approx \frac{-2}{7\pi}$$

$$\vdots$$



Properties of the Dirichlet form: Decibels

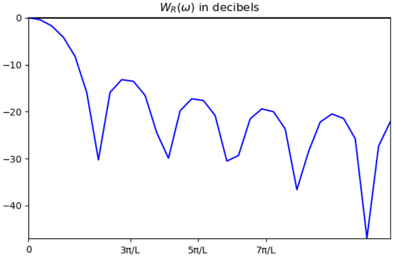
We often describe the relative sidelobe amplitudes in decibels, which are defined as

$$20\log_{10}\left|\frac{W\left(\frac{3\pi}{L}\right)}{W(0)}\right| \approx 20\log_{10}\frac{2}{3\pi} \approx -13\text{dB}$$

$$20\log_{10}\left|\frac{W\left(\frac{5\pi}{L}\right)}{W(0)}\right| \approx 20\log_{10}\frac{2}{5\pi} \approx -18\text{dB}$$

$$20\log_{10}\left|\frac{W\left(\frac{7\pi}{L}\right)}{W(0)}\right| \approx 20\log_{10}\frac{2}{7\pi} \approx -21\text{dB}$$

$$\vdots$$

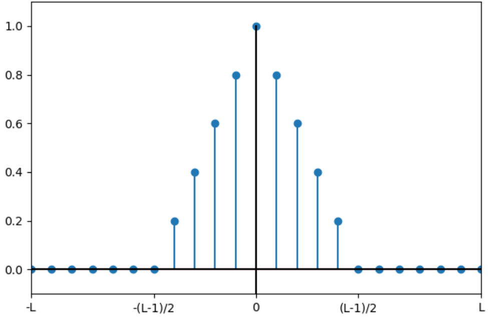


Outline

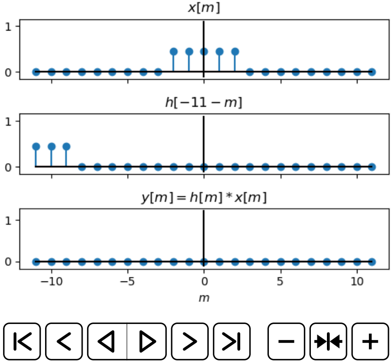
Bartlett (Triangular) Window

A Bartlett window is a triangle:

$$w_B[n] = \max\left(0, 1 - \frac{|n|}{(L-1)/2}\right)$$



A Bartlett window is the convolution of two rectangular windows, each with a height of $\sqrt{\frac{2}{L-1}}$ and a length of $\frac{L-1}{2}$.

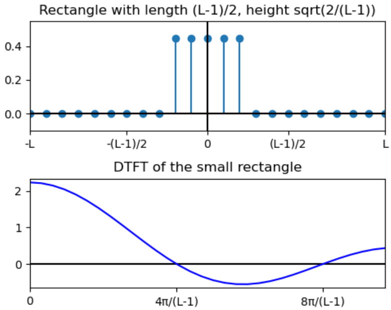


Since each of the two little rectangles has a height of $\sqrt{\frac{2}{L-1}}$ and a length of $\frac{L-1}{2}$, their spectra have a DC value of

$$W_B(0) = \sqrt{\frac{L-1}{2}},$$

and nulls of

$$W_B\left(\frac{4\pi k}{L-1}\right) = 0$$

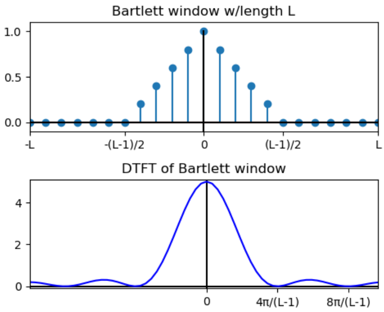


Since

$$w_B[n] = w_R[n] * w_R[n],$$

therefore

$$W_B(\omega) = (W_R(\omega))^2$$

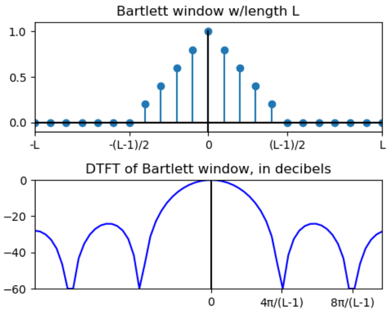


In particular: the sidelobes of a Bartlett window are much lower than those of a rectangular window!

$$20\log_{10}\left|\frac{W_B\left(\frac{6\pi}{L-1}\right)}{W(0)}\right| \approx -26\text{dB}$$

$$20\log_{10}\left|\frac{W\left(\frac{10\pi}{L-1}\right)}{W(0)}\right| \approx -36\text{dB}$$

$$\vdots$$



Things to Notice

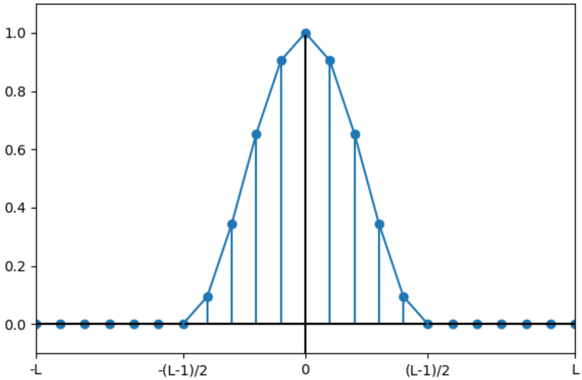
- The **main lobe width** has been doubled, because the Bartlett window is created by convolving two half-length rectangular windows.
  - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have a wider transition band.
- The **sidelobe height** has been dramatically reduced, because convolving in the time domain means multiplying in the frequency domain.
  - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have much lower stopband ripple.

Outline

The Hann Window

Here's the Hann window:

$$w_N[n] = w_R[n]\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n}{L-1}\right)\right)$$



Spectrum of the Hann Window

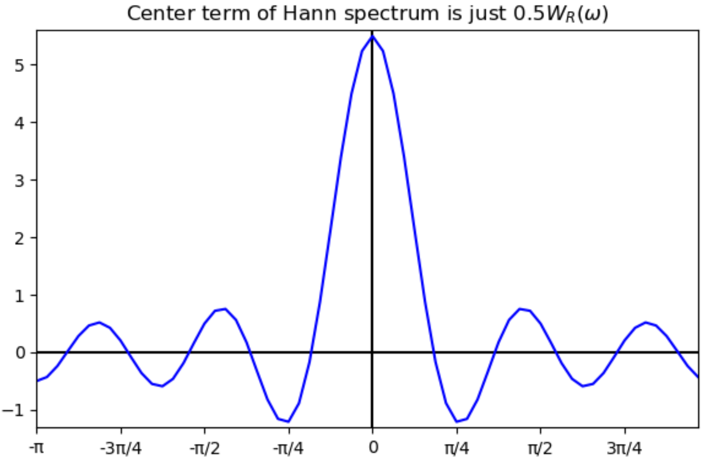
$$w_N[n] = w_R[n]\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n}{L-1}\right)\right)$$
$$= \frac{1}{2}w_R[n] + \frac{1}{4}w_R[n]e^{-j\frac{2\pi}{L-1}} + \frac{1}{4}w_R[n]e^{+j\frac{2\pi}{L-1}}$$

So its spectrum is:

$$W_N(\omega) = \frac{1}{2}W_R(\omega) + \frac{1}{4}W_R\left(\omega - \frac{2\pi}{L-1}\right) + \frac{1}{4}W_R\left(\omega + \frac{2\pi}{L-1}\right)$$

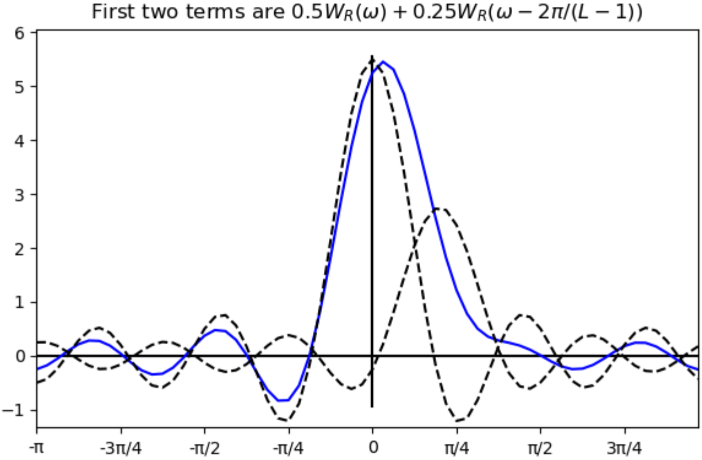
Spectrum of the Rectangular Window

Here's the DTFT of the rectangular window, $0.5W_R(\omega)$:



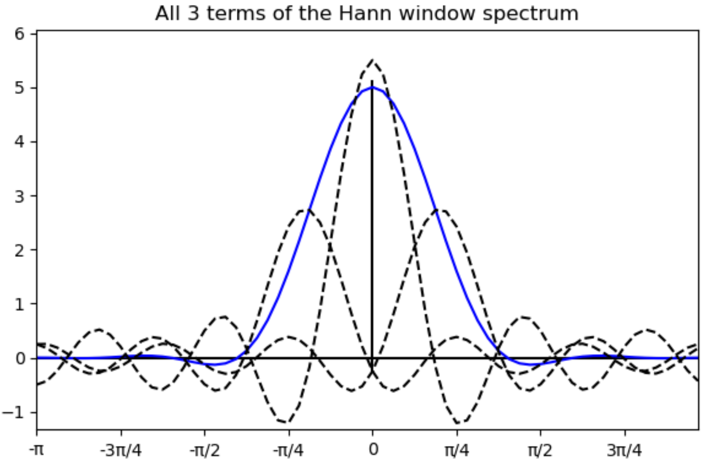
Spectrum of Two Parts of the Hann Window

Here's the DTFT of two parts of the Hann Window,
$\frac{1}{2}W_R(\omega)+\frac{1}{4}W_R\left(\omega-\frac{2\pi}{L-1}\right)$:



Spectrum of the Hann Window

Here's the DTFT of the Hann window,
$W_N(\omega) = \frac{1}{2}W_R(\omega) + \frac{1}{4}W_R\left(\omega - \frac{2\pi}{L-1}\right) + \frac{1}{4}W_R\left(\omega + \frac{2\pi}{L-1}\right)$:



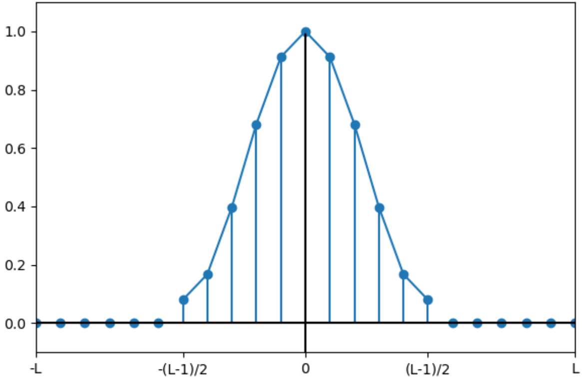
Things to Notice

- The **main lobe width** has been doubled, because each of the two nulls next to the main lobe have been canceled out.
    - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have a wider transition band.
- The **sidelobe height** has been dramatically reduced, because the frequency-shifted copies each cancel out the main copy.
    - Therefore $H(\omega) = \frac{1}{2\pi} W_N(\omega) * H_i(\omega)$ will have much lower stopband ripple.

The Hamming Window

Here's the Hamming window:

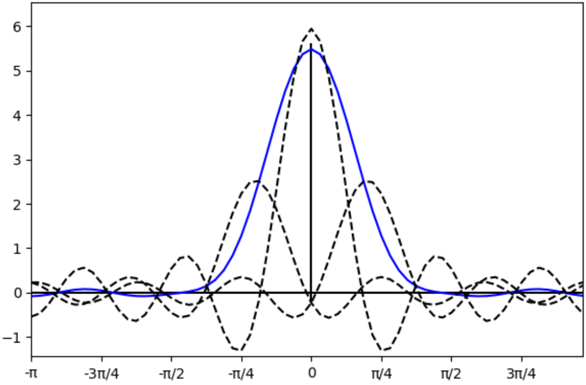
$$w_M[n] = w_R[n]\left(A + (1-A)\cos\left(\frac{2\pi n}{L-1}\right)\right)$$



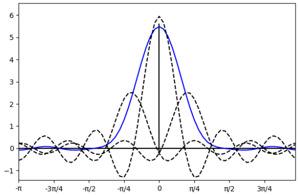
Spectrum of the Hamming Window

$$W_M(\omega) = AW_R(\omega) + \frac{1-A}{2} W_R\left(\omega - \frac{2\pi}{L-1}\right) + \frac{1-A}{2} W_R\left(\omega + \frac{2\pi}{L-1}\right),$$

where A is chosen to minimize the height of the first sidelobe:



Spectrum of the Hamming Window



The first sidelobe is at $\omega = \frac{5\pi}{L}$. At that frequency, $W_M(\omega)$ is roughly:

$$AW_R\left(\frac{5\pi}{L}\right) + \frac{1-A}{2}W_R\left(\frac{5\pi}{L} - \frac{2\pi}{L}\right) + \frac{1-A}{2}W_R\left(\frac{5\pi}{L} + \frac{2\pi}{L}\right)$$

$$\approx A\left(\frac{L}{5\pi}\right) - \frac{1-A}{2}\left(\frac{L}{3\pi}\right) - \frac{1-A}{2}\left(\frac{L}{7\pi}\right)$$

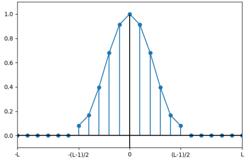
$$\approx (0.13945A - 0.07579)\,L,$$

. . . which is zero if $A = 0.5434782$.

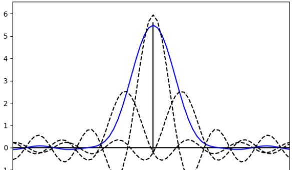
The Hamming Window

The Hamming window chooses $A = 0.5434782$, rounded off to two significant figures:

$$w_M[n] = w_R[n]\left(0.54 + 0.46\cos\left(\frac{2\pi n}{L-1}\right)\right)$$



...with the result that the first sidelobe of the Hamming window has an amplitude below 0.01:



Outline

Main Features of Four Windows

Window	Shape	First Null ($\approx$ Transition Bandwidth)	First Sidelobe ($\approx$ Stopband Ripple)	First Sidelobe Level
Rectangular	rectangle	$\frac{2\pi}{L}$	$0.11L$	-13dB
Bartlett	triangle	$\frac{4\pi}{L}$	$0.05L$	-26dB
Hann	raised cosine	$\frac{4\pi}{L}$	$-0.028L$	-31dB
Hamming	raised cosine	$\frac{4\pi}{L}$	$-0.0071L$	-43dB