Lecture 19: Windowing

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ECE 401: Signal and Image Analysis, Fall 2023

Outline

Review: Ideal Filters

- Ideal Lowpass Filter:

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c, \\ 0 & \omega_c < |\omega| \le \pi. \end{cases} \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Highpass Filter:

$$H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Bandpass Filter:

$$\begin{aligned} H_{BP}(\omega) &= H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega) \\ \leftrightarrow h_{BP}[n] &= \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n) \end{aligned}$$

Outline

Ideal Filters are Infinitely Long

- All of the ideal filters, $h_{LP,i}[n]$ and so on, are infinitely long!
- In demos so far, I've faked infinite length by just making $h_{LP,i}[n]$ more than twice as long as $x[n]$.
- If $x[n]$ is very long (say, a 24-hour audio recording), you probably don't want to do that (computation=expensive)

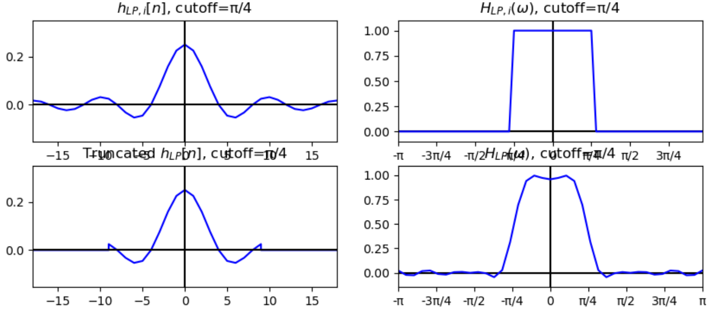
Finite Length by Truncation

We can force $h_{LP,i}[n]$ to be finite length by just truncating it, say, to $2M+1$ samples:

$$h_{LP}[n] = \begin{cases} h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

Truncation Causes Frequency Artifacts

The problem with truncation is that it causes artifacts.



Windowing Reduces the Artifacts

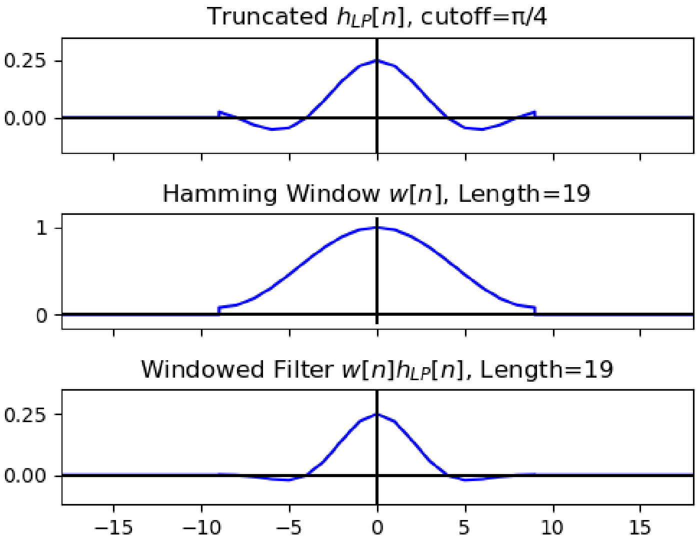
We can reduce the artifacts (a lot) by windowing $h_{LP,i}[n]$, instead of just truncating it:

$$h_{LP}[n] = \begin{cases} w[n]h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

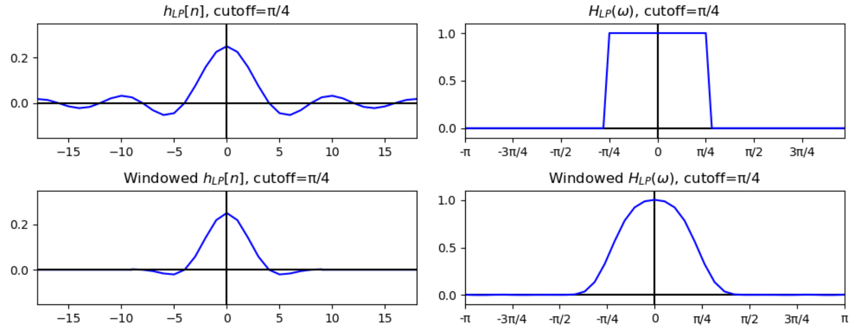
where $w[n]$ is a window that tapers smoothly down to near zero at $n = \pm M$, e.g., a Hamming window:

$$w[n] = 0.54 + 0.46 \cos\left(\frac{2\pi n}{2M}\right)$$

Windowing a Lowpass Filter



Windowing Reduces the Artifacts



Outline

Why does truncation cause artifacts?

But why does truncation cause artifacts?
The reason is that, when we truncate an impulse response, we are (uintentionally?) multiplying it by a rectangular window:

$$h_{LP}[n] = \begin{cases} h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$
$$= w_R[n] h_{LP,i}[n]$$

...where $w_R[n]$ is a function called the "rectangular window:"

$$w_R[n] = \begin{cases} 1 & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

Review: DTFT of Convolution is Multiplication

Remember that the DTFT of convolution is multiplication. If

$$y[n] = h[n] * x[n]$$

. . . then . . .

$$Y(\omega) = H(\omega)X(\omega)$$

New Stuff: DTFT of Multiplication is Convolution!

Guess what: the DTFT of multiplication is ($1/2\pi$ times) convolution!! If

$$g[n] = w[n]h[n]$$

…then …

$$G(\omega) = \frac{1}{2\pi} W(\omega) * H(\omega)$$

Definition and proof: convolution in frequency

The previous slide used the formula "$W(\omega) * H(\omega)$". What does that even mean?

To find out, let's try taking the DTFT of $g[n]$:

$$\begin{aligned}
G(\omega) &= \sum_n g[n] e^{-j\omega n} \\
&= \sum_n w[n] h[n] e^{-j\omega n} \\
&= \sum_n w[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) e^{j\theta n} d\theta \right) e^{-j\omega n}
\end{aligned}$$

In the last line, notice the difference between θ and ω. One is the dummy variable for the IDTFT, one is the dummy variable for the DTFT.

Definition and proof: convolution in frequency

Now let's complete the derivation:

$$G(\omega) = \sum_n w[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) e^{j\theta n} d\theta \right) e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) \left(\sum_n w[n] e^{-j(\omega-\theta)n} \right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) W(\omega-\theta) d\theta$$

New Stuff: DTFT of Multiplication is Convolution!

So when we window a signal in the time domain,

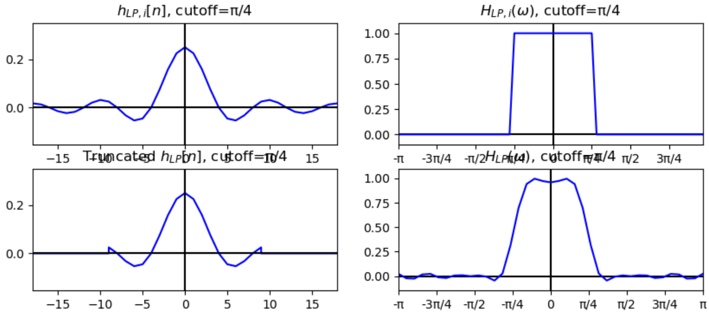
$$g[n] = w[n]h[n]$$

That's equivalent to convolving $H(\omega)$ by the DTFT of the window,

$$\begin{aligned} G(\omega) &= \frac{1}{2\pi} W(\omega) * H(\omega) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) W(\omega - \theta) d\theta \end{aligned}$$

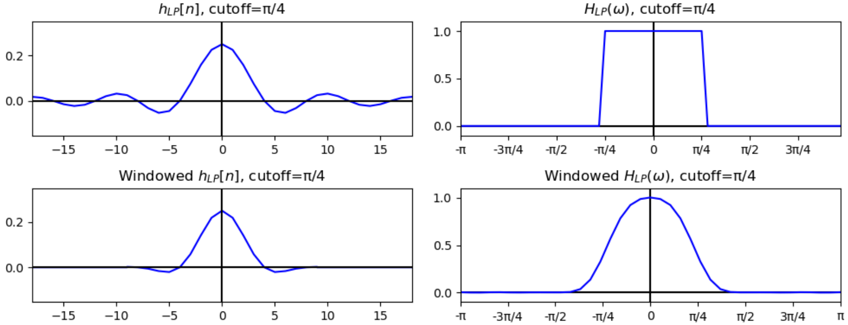
Windowing Causes Frequency Artifacts

We've already seen the result. Windowing by a rectangular window (i.e., truncation) causes nasty artifacts!



Windowing Reduces the Artifacts

. . . whereas windowing by a smooth window, like a Hamming window, causes a lot less artifacts:



Outline

Even Length Filters

Often, we'd like our filter $h_{LP}[n]$ to be even length, e.g., 200 samples long, or 256 samples. We can't do that with this definition:

$$h_{LP}[n] = \begin{cases} w[n]h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

...because $2M+1$ is always an odd number.

Even Length Filters using Delay

We can solve this problem using the time-shift property of the DTFT:

$$z[n] = x[n - n_0] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega n_0} X(\omega)$$

Even Length Filters using Delay

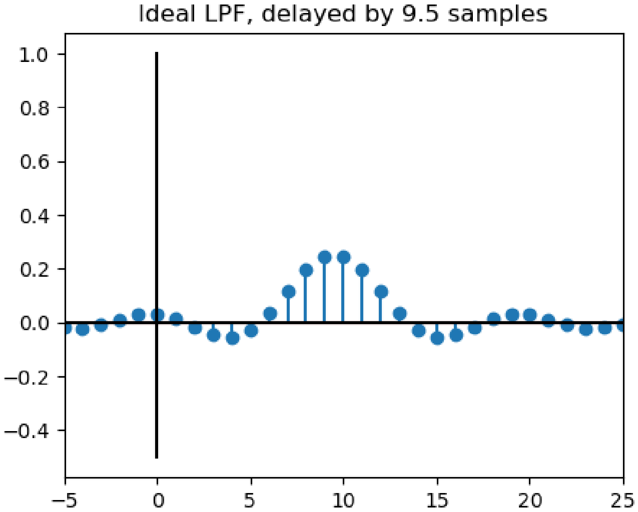
Let's delay the ideal filter by exactly $M-0.5$ samples, for any integer M:

$$z[n]=h_{LP,i}\left[n-(M-0.5)\right]=\frac{\omega_c}{\pi}\text{sinc}\left(\omega\left(n-M+\frac{1}{2}\right)\right)$$

I know that sounds weird. But notice the symmetry it gives us. The whole signal is symmetric w.r.t. sample $n=M-0.5$. So $z[M-1]=z[M]$, and $z[M-2]=z[M+1]$, and so one, all the way out to

$$z[0]=z[2M-1]=\frac{\omega_c}{\pi}\text{sinc}\left(\omega\left(M-\frac{1}{2}\right)\right)$$

Even Length Filters using Delay



Even Length Filters using Delay

Apply the time delay property:

$$z[n] = h_{LP,i}\left[n - (M - 0.5)\right] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega(M-0.5)} H_{LP,i}(\omega),$$

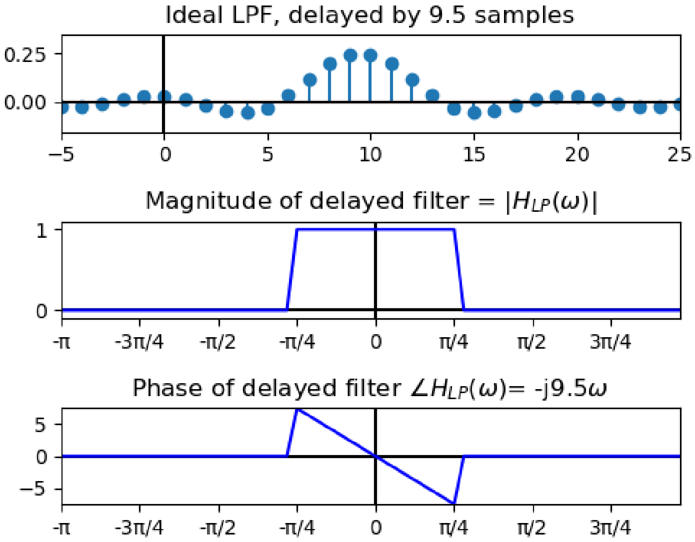
and then notice that

$$\left|e^{-j\omega(M-0.5)}\right| = 1$$

So

$$|Z(\omega)| = |H_{LP,i}(\omega)|$$

Even Length Filters using Delay



Even Length Filters using Delay and Windowing

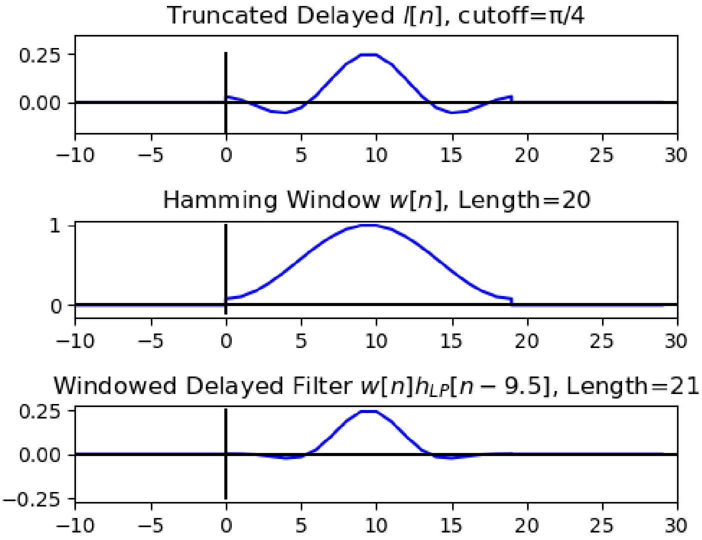
Now we can create an even-length filter by windowing the delayed filter:

$$h_{LP}[n] = \begin{cases} w[n] h_{LP,i}\left[n - (M - 0.5)\right] & 0 \le n \le (2M - 1) \\ 0 & \text{otherwise} \end{cases}$$

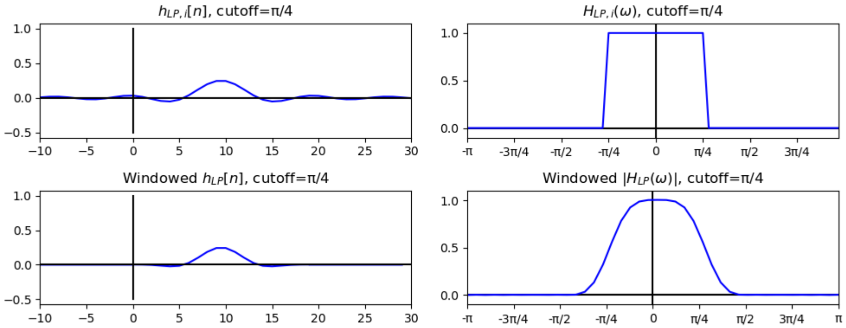
where $w[n]$ is a Hamming window defined for the samples $0 \le m \le 2M - 1$:

$$w[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{2M - 1}\right)$$

Even Length Filters using Delay and Windowing



Even Length Filters using Delay and Windowing



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Summary: Ideal Filters

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Summary: Practical Filters

- Odd Length:

$$h_{HP}[n] = \begin{cases} h_{HP,i}[n]w[n] & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Even Length:

$$h_{HP}[n] = \begin{cases} h_{HP,i}\left[n-(M-0.5)\right]w[n] & 0 \le n \le 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

where $w[n]$ is a window with tapered ends, e.g.,

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{L-1}\right) & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Outline

Written Example

Design a bandpass filter with lower and upper cutoffs of $\omega_1 = \frac{\pi}{3}$, $\omega_2 = \frac{\pi}{2}$, and with a length of $N = 33$ samples, using a Hamming window.