Lecture 18: Ideal Filters

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2023

Outline

Review: DTFT

The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$

$$g[n] = \delta[n-n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X(\omega + 2\pi) = X(\omega)$
1. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

2. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
3. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
4. Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Outline

What is “Ideal”?

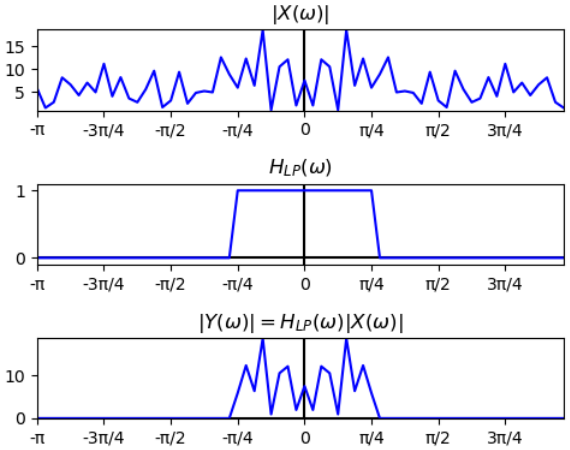
The definition of “ideal” depends on your application. Let’s start with the task of lowpass filtering. Let’s define an ideal lowpass filter, $Y(\omega) = H_{LP}(\omega)X(\omega)$, as follows:

$$Y(\omega) = \begin{cases} X(\omega) & |\omega| \le \omega_c, \\ 0 & \text{otherwise}, \end{cases}$$

where ω_c is some cutoff frequency that we choose. For example, to de-noise a speech signal we might choose $\omega_c = 2\pi 2400/F_s$, because most speech energy is below 2400Hz. This definition gives:

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Ideal Lowpass Filter



How can we implement an ideal LPF?

1. Use `np.fft.fft` to find $X[k]$, set $Y[k] = X[k]$ only for $\frac{2\pi k}{N} < \omega_c$, then use `np.fft.ifft` to convert back into the time domain?
   - It sounds easy, but...
   - `np.fft.fft` is finite length, whereas the DTFT is infinite length. Truncation to finite length causes artifacts.
2. Use pencil and paper to inverse DTFT $H_{LP}(\omega)$ to $h_{LP}[n]$, then use `np.convolve` to convolve $h_{LP}[n]$ with $x[n]$.
   - It sounds more difficult.
   - But actually, we only need to find $h_{LP}[n]$ once, and then we'll be able to use the same formula for ever afterward.
   - This method turns out to be both easier and more effective in practice.

Inverse DTFT of $H_{LP}(\omega)$

The ideal LPF is

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

The inverse DTFT is

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(\omega) e^{j\omega n} d\omega$$

Combining those two equations gives

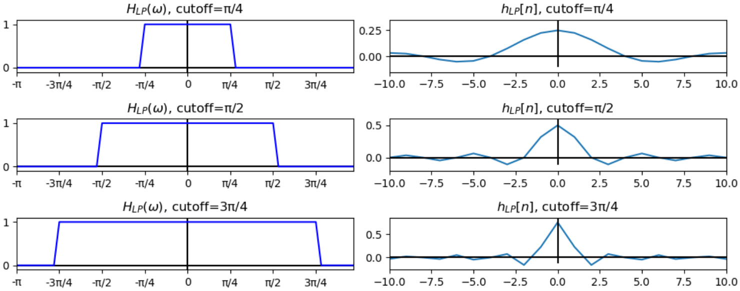
$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

Solving the integral

The ideal LPF is

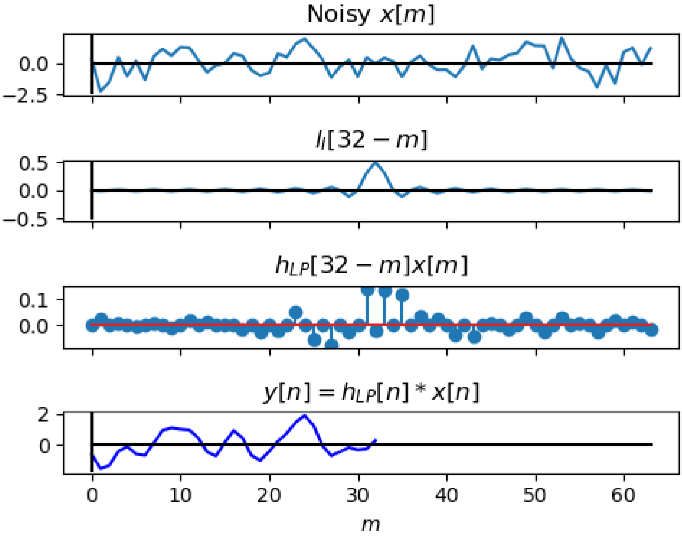
$$
\begin{aligned}
h_{LP}[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi}\left(\frac{1}{jn}\right)\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} \\
&= \frac{1}{2\pi}\left(\frac{1}{jn}\right)\left(2j\sin(\omega_c n)\right) \\
&= \frac{\sin(\omega_c n)}{\pi n} \\
&= \left(\frac{\omega_c}{\pi}\right)\text{sinc}(\omega_c n)
\end{aligned}
$$

$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}$



- $\frac{\sin(\omega_c n)}{\pi n}$ is undefined when $n = 0$
- $\lim_{n\to 0} \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}$
- So let's define $h_{LP}[0] = \frac{\omega_c}{\pi}$.

$h_{LP}[n] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$



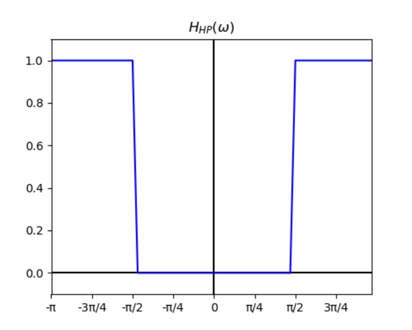
Outline

Ideal Highpass Filter

An ideal high-pass filter passes all frequencies above ω_c:

$$H_{HP}(\omega) = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & \text{otherwise} \end{cases}$$

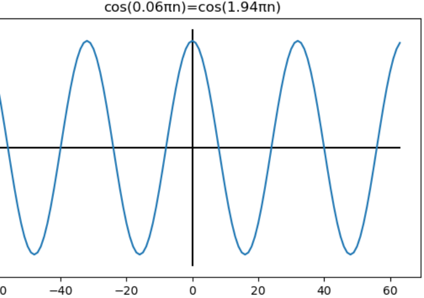
Ideal Highpass Filter

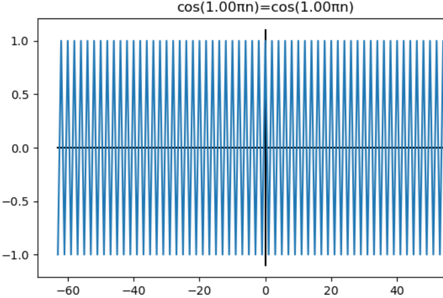


Ideal Highpass Filter

. . . except for one problem: aliasing.

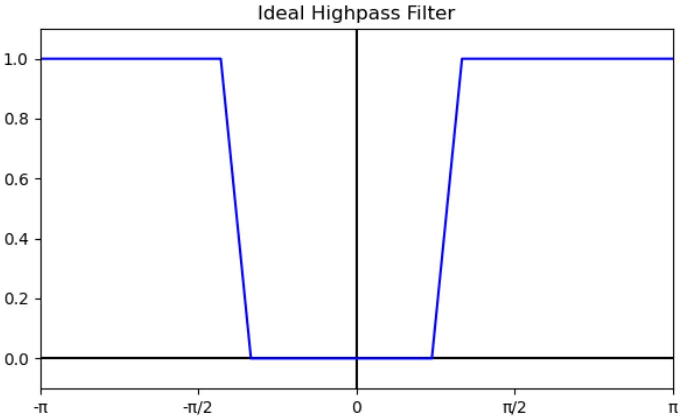
The highest frequency, in discrete time, is $\omega = \pi$. Frequencies that seem higher, like $\omega = 1.1\pi$, are actually lower. This phenomenon is called "aliasing."





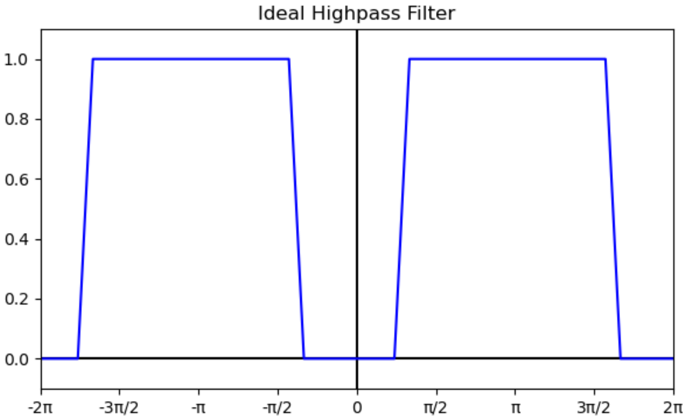
Ideal Highpass Filter

Here's how an ideal HPF looks if we only plot from $-\pi \leq \omega \leq \pi$:



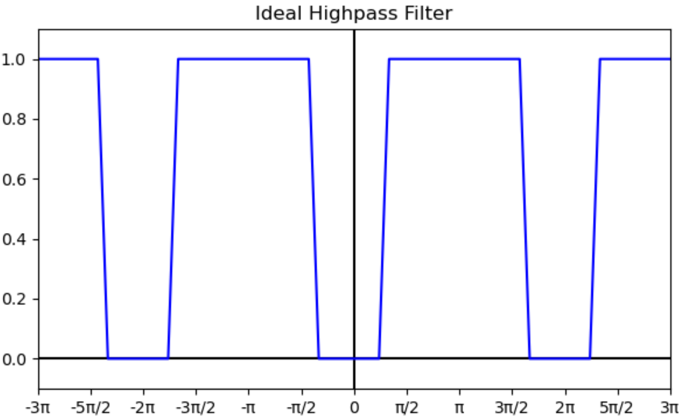
Ideal Highpass Filter

Here's how an ideal HPF looks if we plot from $-2\pi \leq \omega \leq 2\pi$:



Ideal Highpass Filter

Here's how an ideal HPF looks if we plot from $-3\pi \leq \omega \leq 3\pi$:



Redefining "Lowpass" and "Highpass"

Let's redefine "lowpass" and "highpass." The ideal LPF is

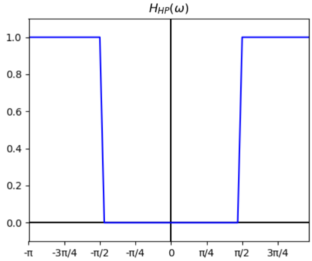
$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c, \\ 0 & \omega_c < |\omega| \le \pi. \end{cases}$$

The ideal HPF is

$$H_{HP}(\omega) = \begin{cases} 0 & |\omega| < \omega_c, \\ 1 & \omega_c \le |\omega| \le \pi. \end{cases}$$

Both of them are periodic with period 2π.

Inverse DTFT of $H_{HP}(\omega)$



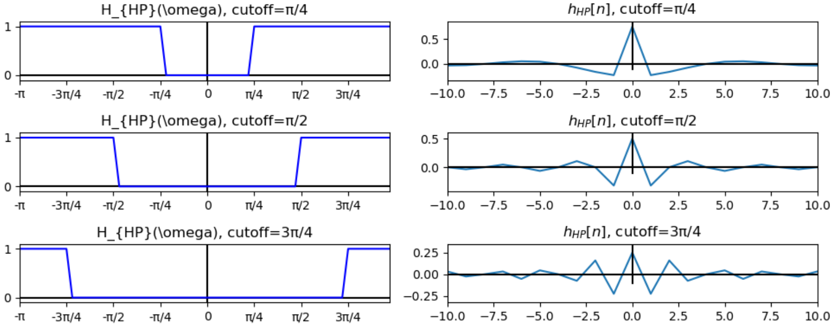
The easiest way to find $h_{HP}[n]$ is to use linearity:

$$H_{HP}(\omega) = 1 - H_{LP}(\omega)$$

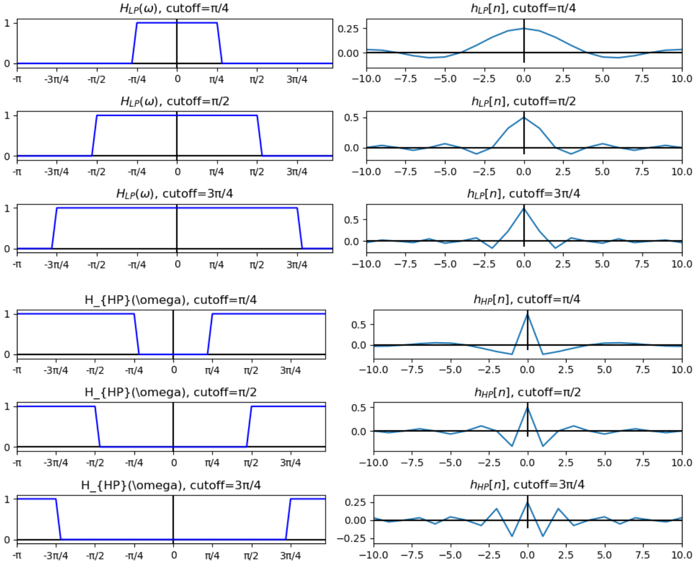
Therefore:

$$\begin{aligned} h_{HP}[n] &= \delta[n] - h_{LP}[n] \\ &= \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n) \end{aligned}$$

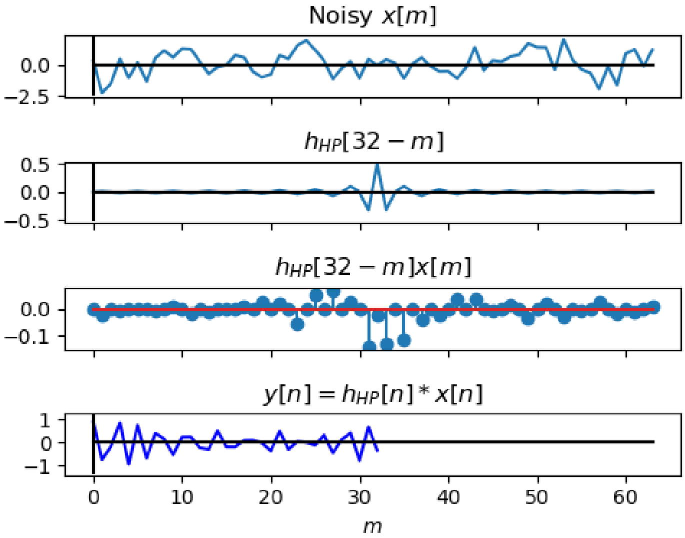
$h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$



Comparing highpass and lowpass filters



$h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$



Outline

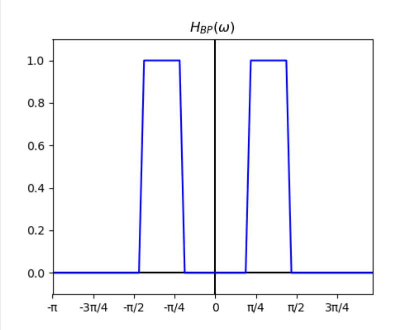
Ideal Bandpass Filter

An ideal band-pass filter passes all frequencies between ω_1 and ω_2:

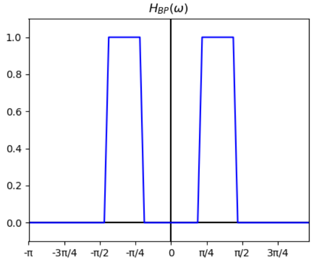
$$H_{BP}(\omega) = \begin{cases} 1 & \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

(and, of course, it's also periodic with period 2π).

Ideal Bandpass Filter



Inverse DTFT of $H_{BP}(\omega)$



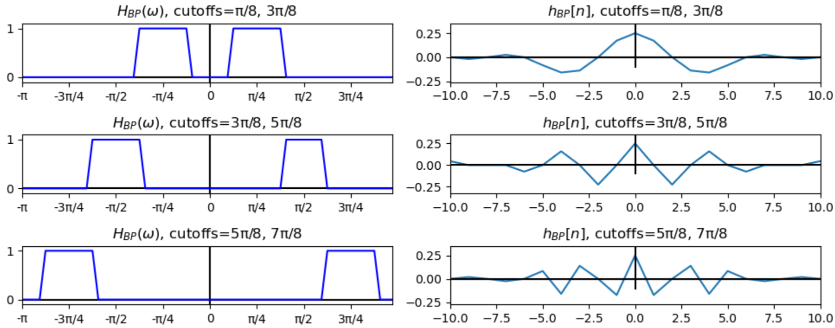
The easiest way to find $h_{BP}[n]$ is to use linearity:

$$H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega)$$

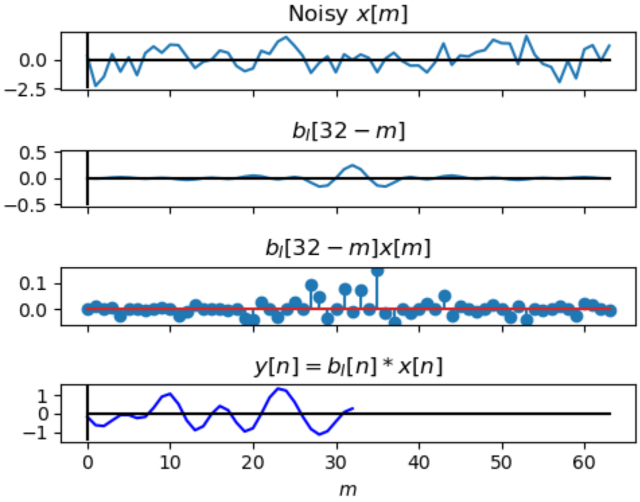
Therefore:

$$h_{BP}[n] = \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n)$$

$h_{BP}[n] = \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n)$



$h_{BP}[n] = \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n)$



Outline

Summary: Ideal Filters

- Ideal Lowpass Filter:

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c, \\ 0 & \omega_c < |\omega| \le \pi. \end{cases} \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Highpass Filter:

$$H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Bandpass Filter:

$$\begin{aligned} H_{BP}(\omega) &= H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega) \\ \leftrightarrow h_{BP}[n] &= \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n) \end{aligned}$$

Outline

Written Example

Suppose you have an image with a sharp boundary, between black and white, at the location $n = 0$. This is well modeled by setting $x[n]$ equal to the unit step function:

$$x[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Use graphical convolution to convolve $x[n]$ with an ideal LPF. You don't need to find the exact values of $y[n]$, but sketch things like: how wide is the ramp between light and dark? How frequent are the ripples on either side of the ramp?