

# Lecture 15: Discrete-Time Fourier Transform (DTFT)

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- 1 Review: Frequency Response
- 2 Discrete Time Fourier Transform
- 3 Properties of the DTFT
- 4 Examples
- 5 Summary
- 6 Written Example

# Outline

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- 6 Written Example

# Response of LSI System to Periodic Inputs

Suppose we compute  $y[n] = x[n] * h[n]$ , where

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \text{ and}$$
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] e^{j2\pi kn/N}.$$

The relationship between  $Y[k]$  and  $X[k]$  is given by the frequency response:

$$Y[k] = H(k\omega_0)X[k]$$

where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

# Response of LSI System to Aperiodic Inputs

But what about signals that never repeat themselves?  
Can we still write something like

$$Y(\omega) = H(\omega)X(\omega)?$$

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# Aperiodic

An “aperiodic signal” is a signal that is not periodic.

- Music: strings, woodwinds, and brass are periodic, drums and rain sticks are aperiodic.
- Speech: vowels and nasals are periodic, plosives and fricatives are aperiodic.
- Images: stripes are periodic, clouds are aperiodic.
- Bioelectricity: heartbeat is periodic, muscle contractions are aperiodic.

# Periodic

The spectrum of a periodic signal is given by its Fourier series. In discrete time, that's:

$$\begin{aligned} X_k &= \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\frac{2\pi kn}{N_0}} \\ &= \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\omega n} \end{aligned}$$

and that gives the frequency content of the signal, at the frequency  $\omega = \frac{2\pi k}{N_0}$ .

Here I'm using  $n \in \left\{ -\frac{N_0}{2}, \dots, \frac{N_0-1}{2} \right\}$ , but the sum could be over any sequence of  $N_0$  continuous samples.



# Aperiodic

An aperiodic signal is one that **never** repeats itself. So we want something like the limit, as  $N_0 \rightarrow \infty$ , of the Fourier series. Here is the simplest such thing that is useful:

## Discrete-Time Fourier Transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

# Fourier Series vs. Fourier Transform

The Fourier Series coefficients are:

$$X_k = \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\omega n}$$

The Fourier transform is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Notice that, besides taking the limit as  $N_0 \rightarrow \infty$ , we also got rid of the  $\frac{1}{N_0}$  factor. So we can think of the DTFT as

$$X(\omega) = \lim_{N_0 \rightarrow \infty, \omega = \frac{2\pi k}{N_0}} N_0 X_k$$

where the limit is: as  $N_0 \rightarrow \infty$ , and  $k \rightarrow \infty$ , but  $\omega = \frac{2\pi k}{N_0}$  remains constant.

# Inverse DTFT

In order to convert  $X(\omega)$  back to  $x[n]$ , we'll take advantage of orthogonality:

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi & m = n \\ 0 & (m - n) = \text{any nonzero integer} \end{cases}$$

# Inverse DTFT

Taking advantage of orthogonality, we can see that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right) e^{j\omega m} d\omega \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x[n] \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega \\ &= x[m] \end{aligned}$$

# Fourier Series and Fourier Transform

Discrete-Time Fourier Series (DTFS):

$$X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j \frac{2\pi kn}{N_0}}$$

$$x[n] = \sum_{k=0}^{N_0-1} X_k e^{j \frac{2\pi kn}{N_0}}$$

Discrete-Time Fourier Transform (DTFT):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

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# Properties of the DTFT

In order to better understand the DTFT, let's discuss these properties:

- 0 Periodicity
- 1 Linearity
- 2 Time Shift
- 3 Frequency Shift
- 4 Filtering is Convolution

Property #4 is actually the reason why we invented the DTFT in the first place. Before we discuss it, though, let's talk about the others.

# 0. Periodicity

The DTFT is periodic with a period of  $2\pi$ . That's just because  $e^{j2\pi} = 1$ :

$$X(\omega) = \sum_n x[n] e^{-j\omega n}$$

$$X(\omega + 2\pi) = \sum_n x[n] e^{-j(\omega+2\pi)n} = \sum_n x[n] e^{-j\omega n} = X(\omega)$$

$$X(\omega - 2\pi) = \sum_n x[n] e^{-j(\omega-2\pi)n} = \sum_n x[n] e^{-j\omega n} = X(\omega)$$

For example, the inverse DTFT can be defined in two different ways:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$$

Those two integrals are equal because  $X(\omega + 2\pi) = X(\omega)$ .



# 1. Linearity

The DTFT is linear:

$$z[n] = ax[n] + by[n] \quad \leftrightarrow \quad Z(\omega) = aX(\omega) + bY(\omega)$$

**Proof:**

$$\begin{aligned} Z(\omega) &= \sum_n z[n]e^{-j\omega n} \\ &= a \sum_n x[n]e^{-j\omega n} + b \sum_n y[n]e^{-j\omega n} \\ &= aX(\omega) + bY(\omega) \end{aligned}$$

## 2. Time Shift Property

Shifting in time is the same as multiplying by a complex exponential in frequency:

$$z[n] = x[n - n_0] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega n_0} X(\omega)$$

**Proof:**

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_0)} \quad (\text{where } m = n - n_0) \\ &= e^{-j\omega n_0} X(\omega) \end{aligned}$$

### 3. Frequency Shift Property

Shifting in frequency is the same as multiplying by a complex exponential in time:

$$z[n] = x[n]e^{j\omega_0 n} \quad \leftrightarrow \quad Z(\omega) = X(\omega - \omega_0)$$

**Proof:**

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n} e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega - \omega_0)n} \\ &= X(\omega - \omega_0) \end{aligned}$$

## 4. Convolution Property

Convolving in time is the same as multiplying in frequency:

$$y[n] = h[n] * x[n] \quad \leftrightarrow \quad Y(\omega) = H(\omega)X(\omega)$$

**Proof:** Remember that  $y[n] = h[n] * x[n]$  means that  $y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$ . Therefore,

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} h[m]x[n-m] \right) e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (h[m]x[n-m]) e^{-j\omega m} e^{-j\omega(n-m)} \\ &= \left( \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \right) \left( \sum_{(n-m)=-\infty}^{\infty} x[n-m]e^{-j\omega(n-m)} \right) \\ &= H(\omega)X(\omega) \end{aligned}$$

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# Impulse and Delayed Impulse

For our examples today, let's consider different combinations of these three signals:

$$f[n] = \delta[n]$$

$$g[n] = \delta[n - 3]$$

$$h[n] = \delta[n - 6]$$

Remember from last time what these mean:

$$f[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$g[n] = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 1 & n = 6 \\ 0 & \text{otherwise} \end{cases}$$

# DTFT of an Impulse

First, let's find the DTFT of an impulse:

$$f[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$F(\omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\omega n}$$
$$= 1 \times e^{-j\omega 0}$$
$$= 1$$

So we get that  $f[n] = \delta[n] \leftrightarrow F(\omega) = 1$ . That seems like it might be important.

# DTFT of a Delayed Impulse

Second, let's find the DTFT of a delayed impulse:

$$g[n] = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$
$$G(\omega) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$
$$= 1 \times e^{-j\omega 3}$$

So we get that

$$g[n] = \delta[n - 3] \leftrightarrow G(\omega) = e^{-j3\omega}$$

Similarly, we could show that

$$h[n] = \delta[n - 6] \leftrightarrow H(\omega) = e^{-j6\omega}$$



# Impulse and Delayed Impulse

So our signals are:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$

$$g[n] = \delta[n - 3] \leftrightarrow G(\omega) = e^{-3j\omega}$$

$$h[n] = \delta[n - 6] \leftrightarrow H(\omega) = e^{-6j\omega}$$

# Time Shift Property

Notice that

$$\begin{aligned}g[n] &= f[n - 3] \\h[n] &= g[n - 3].\end{aligned}$$

From the time-shift property of the DTFT, we can get that

$$\begin{aligned}G(\omega) &= e^{-j3\omega} F(\omega) \\H(\omega) &= e^{-j3\omega} G(\omega).\end{aligned}$$

Plugging in  $F(\omega) = 1$ , we get

$$\begin{aligned}G(\omega) &= e^{-j3\omega} \\H(\omega) &= e^{-j6\omega},\end{aligned}$$

which we already know to be the right answer!

# Convolution Property and the Impulse

Notice that, if  $F(\omega) = 1$ , then anything times  $F(\omega)$  gives itself again. In particular,

$$G(\omega) = G(\omega)F(\omega)$$

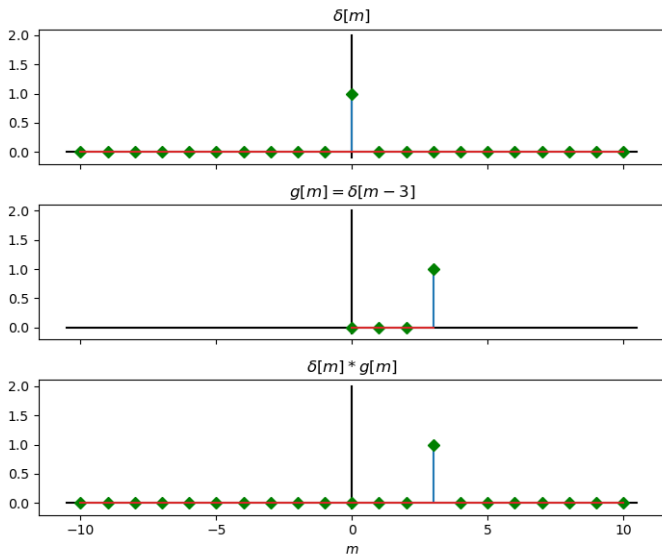
$$H(\omega) = H(\omega)F(\omega)$$

Since multiplication in frequency is the same as convolution in time, that must mean that when you convolve any signal with an impulse, you get the same signal back again:

$$g[n] = g[n] * \delta[n]$$

$$h[n] = h[n] * \delta[n]$$

# Convolution Property and the Impulse



# Convolution Property and the Delayed Impulse

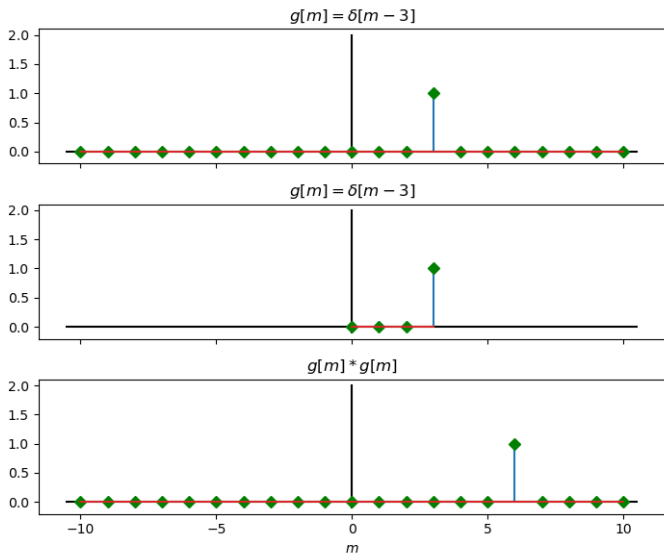
Here's another interesting thing. Notice that  $G(\omega) = e^{-j3\omega}$ , but  $H(\omega) = e^{-j6\omega}$ . So

$$\begin{aligned} H(\omega) &= e^{-j3\omega} e^{-j3\omega} \\ &= G(\omega)G(\omega) \end{aligned}$$

Does that mean that:

$$\delta[n - 6] = \delta[n - 3] * \delta[n - 3]$$

# Convolution Property and the Delayed Impulse



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# Summary

The DTFT (discrete time Fourier transform) of any signal is  $X(\omega)$ , given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n - n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$



# Properties of the DTFT

Properties worth knowing include:

- 0 Periodicity:  $X(\omega + 2\pi) = X(\omega)$
- 1 Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

- 2 Time Shift:  $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
- 3 Frequency Shift:  $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
- 4 Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

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# Written Example

Suppose that  $h[n]$  and  $x[n]$  are identical rectangle functions:

$$x[n] = h[n] = \begin{cases} 1 & -5 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

- 1 Find  $y[n] = h[n] * x[n]$  by calculating the convolution.
- 2 Find  $H(\omega)$ .
- 3 Find  $Y(\omega) = H(\omega)X(\omega)$