Lecture 15: Causality and Stability

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ECE 401: Signal and Image Analysis, Fall 2023

1. Review: Impulse Response and Frequency Response

2. Causality = The future is unknown

3. Stability = All finite inputs produce finite outputs

4. Summary

Outline

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Impulse Response and Convolution

The impulse response of a system is its response to an impulse:

$$\delta[n] \stackrel{\mathcal{H}}{\longrightarrow} h[n]$$

If a system is linear and shift-invariant, then its output, in response to **any** input, can be commputed using convolution:

$$x[n] \stackrel{\mathcal{H}}{\longrightarrow} y[n] = h[n] * x[n]$$

Frequency Response

The frequency response of a system is its response to a pure tone:

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$

$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \angle H(\omega))$$

The frequency response is related to the impulse response by:

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Outline

Causality

Definition: A **causal** system is a system whose output at time n, $y[n]$, depends on inputs $x[m]$ only for $m \leq n$.

What systems are causal? What systems are non-causal?

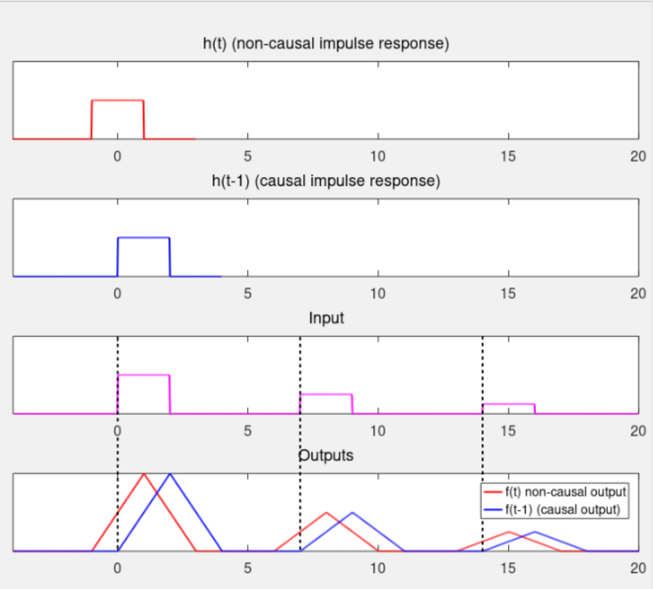
- A real-time system must be causal.
- If n is time, but the system is operating in batch mode, then it doesn't need to be causal.
- If n is space (e.g., rows or columns of an image), the system doesn't need to be causal.

Causal system $\Leftrightarrow$ Right-sided impulse response

$$y[n] = \sum_m h[m]x[n-m]$$

- This system is **causal** iff $y[n]$ depends on $x[n-m]$ only for $n-m \le n$.
- In other words, the system is causal iff $h[n] = 0$ for all $n < 0$.

Causal system ⇔ Right-sided impulse response



Variations on the word "causal"

- A **causal** system is one that depends only on the present and the past, thus $h[n] = 0$ for $n < 0$.
- A **non-causal** system is one that's not causal.
- A **anti-causal** system is one that depends only on the present and the future, thus $h[n] = 0$ for $n > 0$.

Causality $\Leftrightarrow$ Non-Positive Phase Response

If you put a cosine into a system, you get a cosine advanced by $\angle H(\omega)$:

$$\cos(\omega n) \xrightarrow{\mathcal{H}} |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right)$$

- If $\angle H(\omega) > 0$, it means that the output is happening **before** the input!
- It turns out that for any causal system, $\angle H(\omega) \leq 0$

Causality ⇔ Non-Positive Phase Response

Remember how we can calculate $H(\omega)$:

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}$$

- If the system is causal, then the only nonzero terms in that sum are terms with non-positive phase ($e^{-j\omega m}$)
- Therfore causal systems have $\angle H(\omega) \leq 0$

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Stability

Definition: A system is **stable** if and only if **every** bounded $x[n]$ (every signal such that $|x[n]| < \infty$ for all n) produces a bounded output ($|y[n]| < \infty$ for all n).

Why Stability Matters

- If your system is unstable, then every now and then, you'll get an inexplicable bug:
- all of the samples of $y[n]$ will be FLT_MAX!
- That's very hard to debug. If you view it as an image, or listen to it, it will sound like you just didn't generate the samples, so you will be looking for the error in the wrong place!

Magnitude-summable impulse response $\Rightarrow$ Stable system

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Suppose we know that $|x[n]| \leq M$, for some finite M, for all n. Then

$$|y[n]| \leq M \sum_{m=-\infty}^{\infty} |h[m]|$$

So

$$\sum_{m=-\infty}^{\infty} |h[m]| \text{ is finite } \Rightarrow \text{ System is stable}$$

Stable system ⇒ Magnitude-summable impulse response

On the other hand, suppose that

$$\sum_{m=-\infty}^{\infty} |h[m]| = \infty$$

Does that mean that the system is **unstable**? Yes! Yes, it does! Consider the "worst-case" input

$$x[n] = \text{sign}\left(h[-n]\right)$$

Then $y[0]$ is

$$y[0] = \sum_{m=-\infty}^{\infty} h[m]x[-m] = \sum_{m=-\infty}^{\infty} |h[m]| = \infty$$

Example: Weighted Average

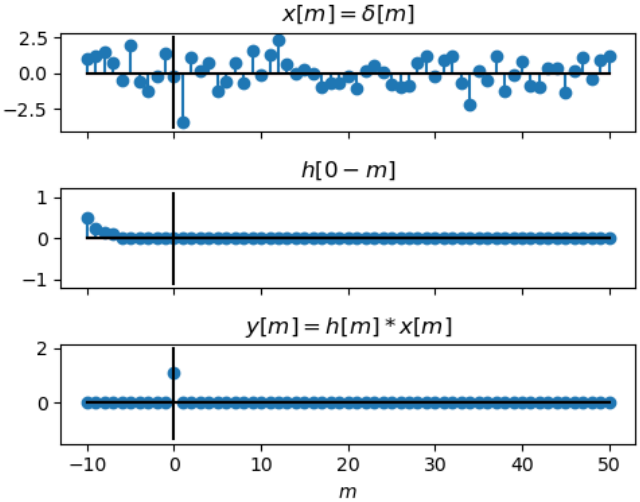
For example, consider a 7-tap weighted average:

$$y[n] = \sum_{m=-3}^{3} h[m]x[n-m]$$

As long as all of the weights are finite ($|h[m]| < \infty$ for all m), then $\sum_{m=-3}^{3} |h[m]|$ is also finite, so the system is stable

Example: Weighted Average

For any finite input, the output is finite:



Example: Summation

For example, consider summation:

$$y[n] = \sum_{m=0}^{\infty} x[n-m]$$

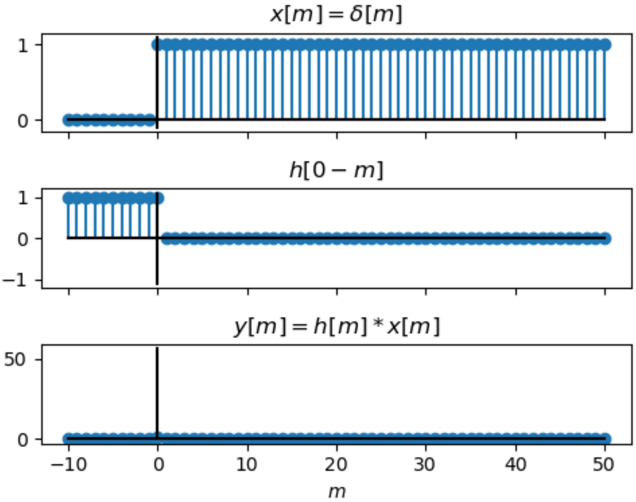
This is an **unstable** system!!

$$h[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 1 = \infty$$

Example: Summation

For example, if the input is a unit step, the output is unbounded:



Example: Obviously Unstable System

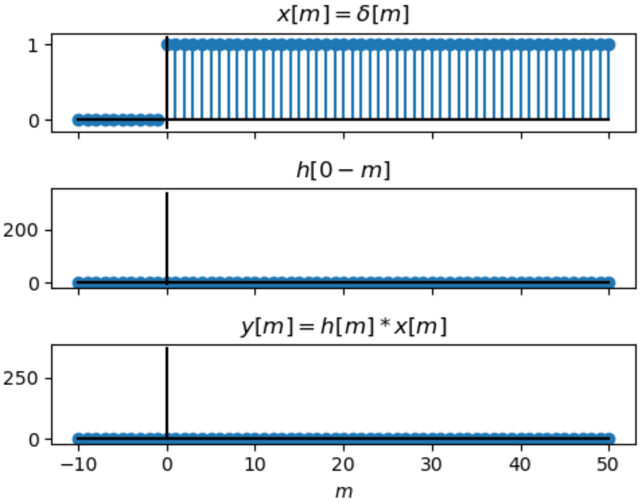
Finally, some systems are just obviously unstable. Consider

$$h[n] = (1.1)^n u[n]$$

This is obviously unstable. In fact, not only does $|h[n]|$ sum to infinity — it even goes to infinity if the input is just a delta function!

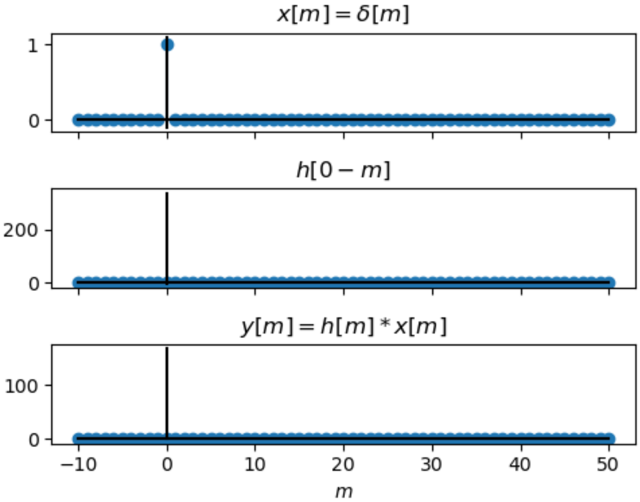
Example: Obviously Unstable System

For example, if the input is a unit step, the output is unbounded:



Example: Obviously Unstable System

Even if the input is a delta function, the output is unbounded:



Relationship to Frequency Response

How about the frequency response of stable versus unstable systems? Guess what:

- A stable system has a finite magnitude response.
- An unstable system usually has an infinite-magnitude (undefined) frequency response.

Proof:

$$|H(\omega)| = \left| \sum_{m=-\infty}^{\infty} h[n] e^{-j\omega n} \right|$$

$$\leq \sum_{m=-\infty}^{\infty} |h[n]|$$

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Summary

- A system is causal if and only if $h[n]$ is right-sided.
  - A causal system has a negative phase response.
- A system is stable if and only if $h[n]$ is magnitude-summable.
  - A stable system has a finite magnitude response.