Lecture 10: DT Filtering of CT Signals

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2023

Outline

Sampling: Continuous Time → Discrete Time

A signal is sampled by measuring its value once every T_s seconds:

$$x[n] = x(t = nT_s)$$

The spectrum of the DT signal has components at $\omega = \frac{2\pi f}{F_s}$, and also at every $2\pi\ell + \omega$ and $2\pi\ell - \omega$, for every integer ℓ. Aliasing occurs unless $|\omega| \leq \pi$.

Interpolation: Discrete Time → Continuous Time

A CT signal $y(t)$ can be created from a DT signal $y[n]$ by interpolation:

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

- $p(t) =$rectangle $\Rightarrow$ PWC interpolation
- $p(t) =$triangle $\Rightarrow$ PWL interpolation
- $p(t) = \text{sinc}\left(\frac{\pi t}{T_s}\right) \Rightarrow$ perfectly bandlimited interpolation, $y(t)$ has no spectral components above F_N

Convolution

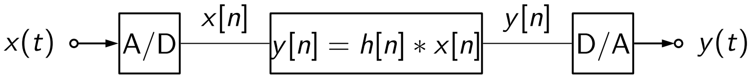
Convolution (finite impulse response filtering) is a generalization of weighted local averaging:

$$y[n] = h[n] * x[n] \equiv \sum_m x[m]h[n-m] = \sum_m x[n-m]h[m]$$

- If all samples of $h[n]$ are positive, then it's a weighted local averaging filter
- If the samples of $h[n]$ are positive for $n > 0$ and negative for $n < 0$ (or vice versa), then it's a weighted local differencing filter

Outline

DT Filtering of CT Signals



Constraints:

- Assume that A/D and D/A use same F_s
- Assume $F_s \geq 2f_{\max}$
- Assume sinc interpolation

DT Filtering of CT Signals

- If $h[n]$ is a local averager, what's the relationship of $y(t)$ to $x(t)$?
- If $h[n]$ is a local differencer, what's the relationship of $y(t)$ to $x(t)$?

Outline

Fourier Series

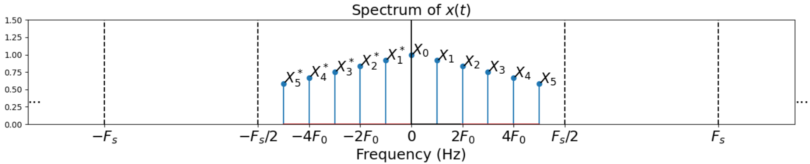
To start with, let's assume $x(t)$ is periodic and bandlimited, so:

$$x(t) = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{j2\pi kF_0 t}$$

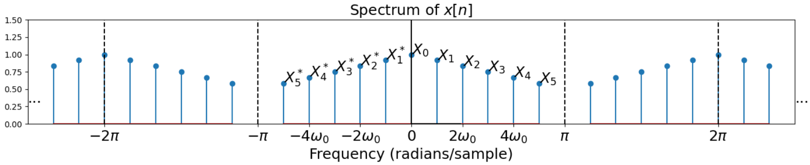
$$x[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{jk\omega_0 n}$$

Where the period $F_s T_0$ might or might not be an integer number of samples, but the signal is bandlimited to $\frac{(N-1)}{2}\omega_0 < \pi$.

Spectrum of $x(t)$



Spectrum of $x[n]$



Relationship of $y[n]$ to $x[n]$

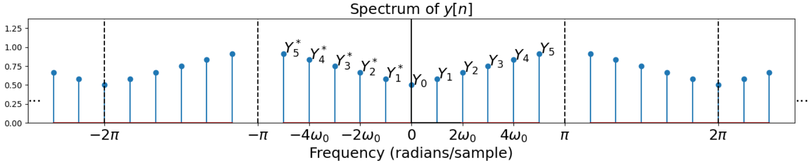
New fact, I haven't yet proven it to you: If the input to a convolution contains only frequencies $k\omega_0$, the output will also only have frequencies $k\omega_0$. Therefore the relationship between $x[n]$ and $y[n]$ is

$$x[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{jk\omega_0 n}$$

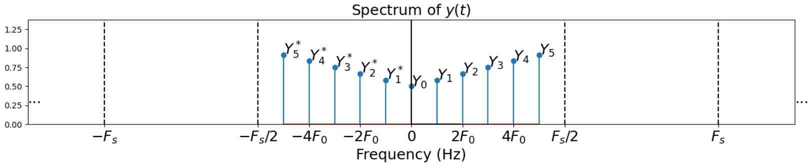
$$y[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} Y_k e^{jk\omega_0 n}$$

The relationship between X_k and Y_k can be almost anything; it depends on $h[n]$ and k and ω_0.

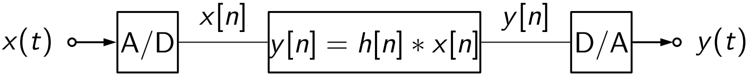
Spectrum of $y[n]$



Spectrum of $y(t)$



DT Filtering of CT Signals



So, from the input to output, the signals are:

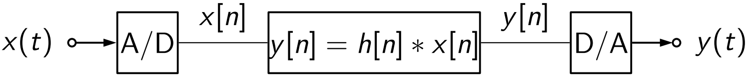
$$x(t) = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{j2\pi kF_0 t}, \qquad x[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{jk\omega_0 n}$$

$$y[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} Y_k e^{jk\omega_0 n}, \qquad y(t) = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} Y_k e^{j2\pi kF_0 t}$$

What is the relationship between Y_k and X_k? Let's solve it in general, then do some examples.

Outline

DT Filtering of a Pure Tone



Let's assume that $x(t)$ is a pure-tone complex exponential at frequency $\omega\left[\frac{\text{radians}}{\text{sample}}\right] = \frac{2\pi\left[\frac{\text{radians}}{\text{cycle}}\right] f\left[\frac{\text{cycles}}{\text{second}}\right]}{F_s\left[\frac{\text{samples}}{\text{second}}\right]}$:

$$x(t) = Xe^{j2\pi ft}$$
$$x[n] = Xe^{j\omega n}$$
$$y[n] = Ye^{j\omega n}$$
$$y(t) = Ye^{j2\pi ft}$$

Can we find the relationship between the two phasors Y and X?

Relationship Between Y and X

Remember that $y[n] = x[n] * h[n]$, i.e.,

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

What happens if we plug in $x[n] = Xe^{j\omega n}$?

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]Xe^{j\omega(n-m)} \\ &= Xe^{j\omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \\ &= Ye^{j\omega n} \end{aligned}$$

Relationship Between Y and X

$$Y = X \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}$$

The sum $\sum h[m]e^{-j\omega m}$ is called the **frequency response** of the system. We'll talk about it a lot over the next few weeks. For now, let's do some examples.

Outline

Rectangular Averager

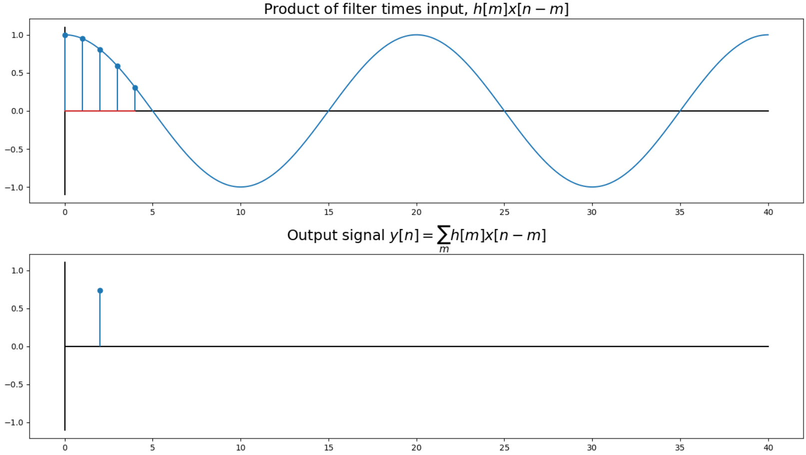
$$y[n] = \sum_m h[m]x[n-m]$$

Consider the case of the rectangular averager:

$$h[n] = \begin{cases} \frac{1}{N} & -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

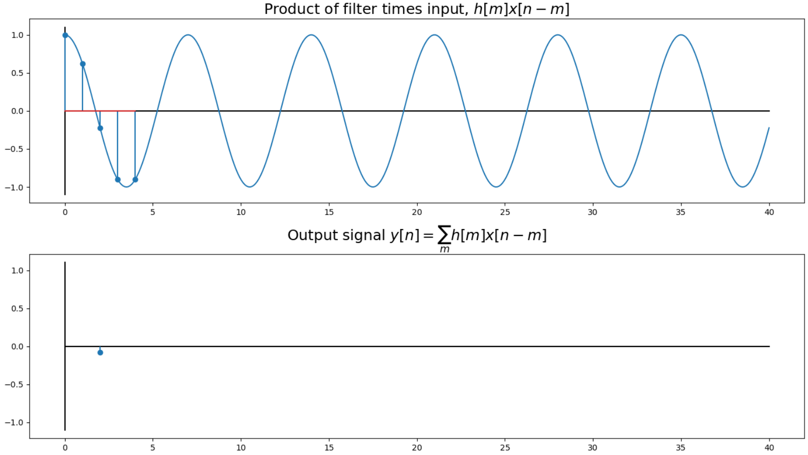
Rectangular Averaging: Low-Frequency Cosine

When the input is low-frequency, the output of an averager is almost the same as the input:



Rectangular Averaging: High-Frequency Cosine

When the input is high-frequency, the system averages over almost one complete period, so the output is close to zero:



Rectangular Averaging: General Case

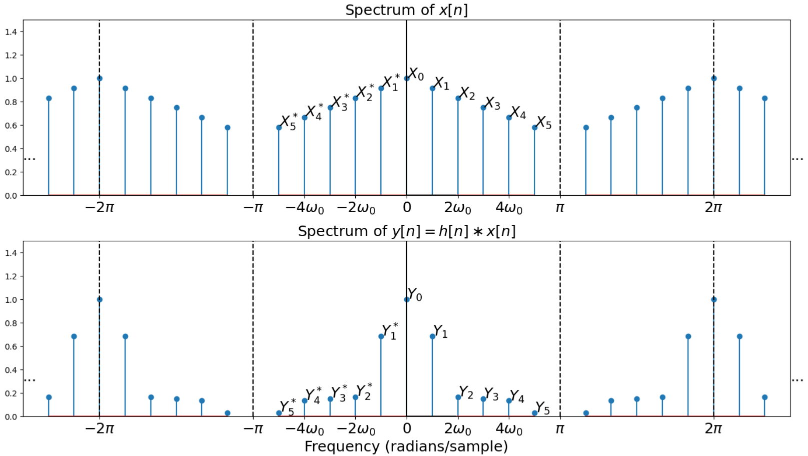
Remember the general form for the frequency response:

$$
\begin{aligned}
Y &= X \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \\
&= \frac{X}{N} \sum_{m=-(N-1)/2}^{(N-1)/2} e^{-j\omega m} \\
&= \frac{X}{N} \left(1 + 2 \sum_{m=1}^{(N-1)/2} \cos(\omega m) \right)
\end{aligned}
$$

- If ω is very small, all terms are positive, so the output is large.
- If ω is larger, then the summation includes both positive and negative terms, so the output is small.

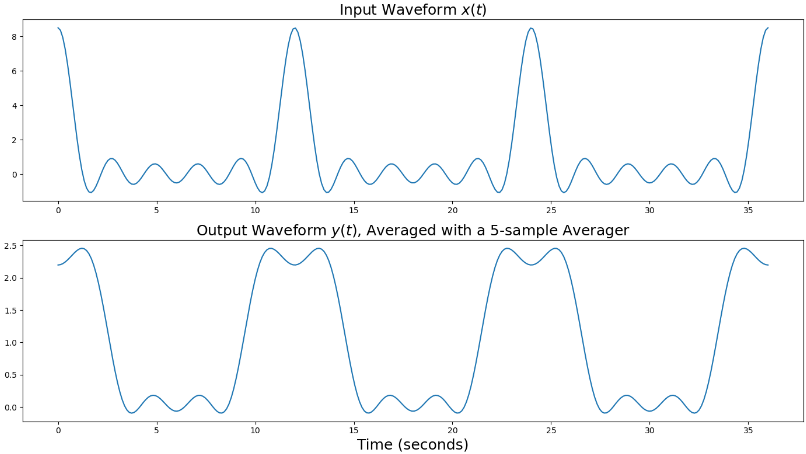
Spectral Plots: Rectangular Averager

The averager retains low-frequency components, but reduces high-frequency components:



Waveforms: Rectangular Averager

The averager tends to smooth out the waveform:



Outline

Binary Differencer

$$y[n] = \sum_m h[m]x[n-m]$$

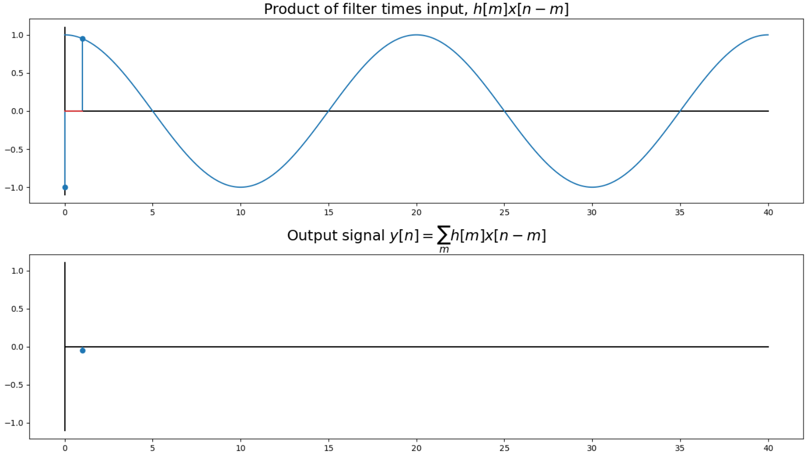
Consider the case of the binary differencer:

$$h[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

. . . so that $y[n] = x[n] - x[n-1]$.

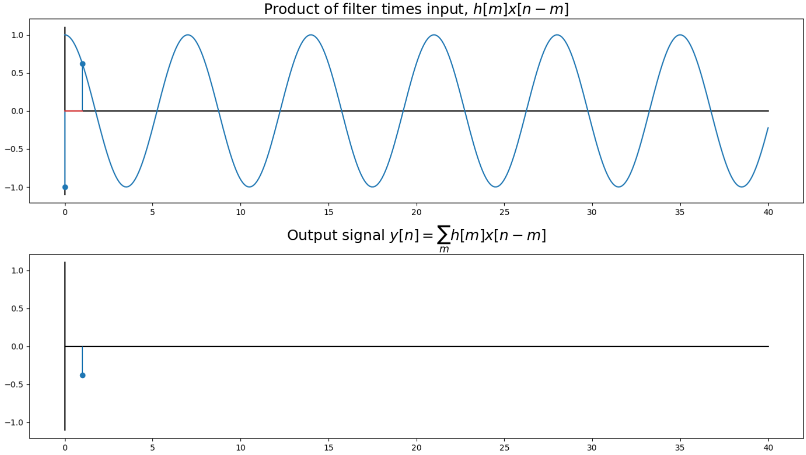
Binary Differencer: Low-Frequency Cosine

When the input is low-frequency, the difference between neighboring samples is nearly zero:



Binary Differencer: High-Frequency Cosine

When the input is high-frequency, the difference between neighboring samples is large:



Binary Differencer: General Case

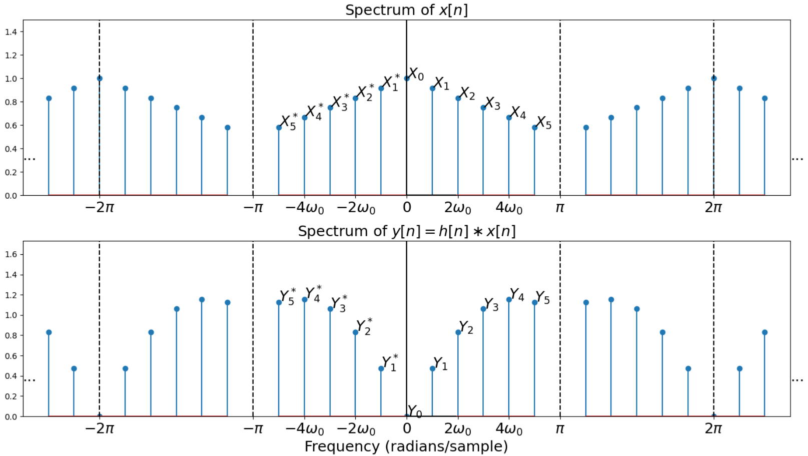
Remember the general form for the frequency response:

$$
\begin{aligned}
Y &= X \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \\
&= X \left(1 - e^{-j\omega}\right) \\
&= X \left(e^{j\omega/2} - e^{-j\omega/2}\right) e^{-j\omega/2} \\
&= 2jX \sin\left(\frac{\omega}{2}\right) e^{-j\omega/2}
\end{aligned}
$$

- If ω is very small, $\sin(\omega/2)$ is very small
- As $\omega \to \pi$ (high frequencies), $\sin(\omega/2) \to 1$

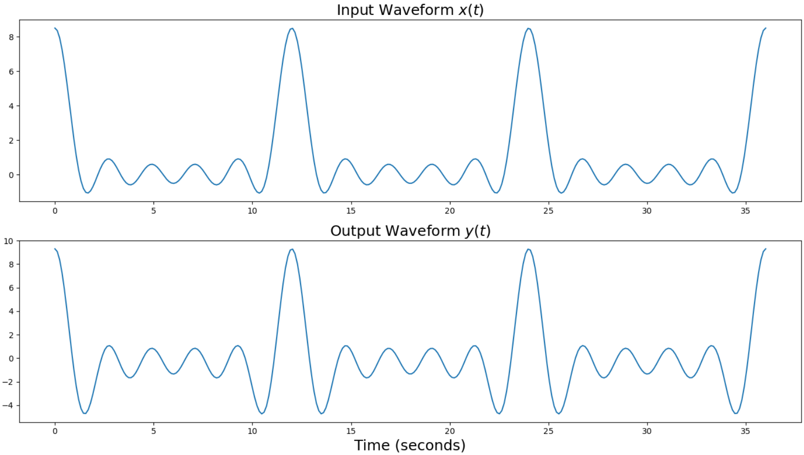
Spectral Plots: Binary Differencer

The binary differencer removes the 0Hz component, but keeps high frequencies:



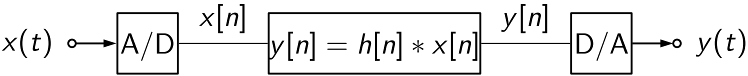
Waveforms: Binary Differencer

The binary differencer removes the 0Hz component, and tends to emphasize “edges” in the waveform:



Outline

Conclusions



If $x(t)$ is periodic, then $y(t)$ is also periodic with the same period but different Fourier Series coefficients:

$$x(t) = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{j2\pi kF_0 t}, \qquad x[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} X_k e^{jk\omega_0 n}$$

$$y[n] = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} Y_k e^{jk\omega_0 n}, \qquad y(t) = \sum_{k=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} Y_k e^{j2\pi kF_0 t}$$

Conclusions

The relationship between the Fourier series coefficients is given by the frequency response of the system:

$$Y = X \sum_m h[m] e^{-j\omega m}$$

- A rectangular averager is a low-pass filter: low-frequency signals pass through, but high-frequency signals are averaged out.
- A binary differencer is a high-pass filter: high-frequency signals pass through, but low-frequency signals are differenced out, especially the 0Hz component.