Lecture 9: Convolution

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ECE 401: Signal and Image Analysis, Fall 2023

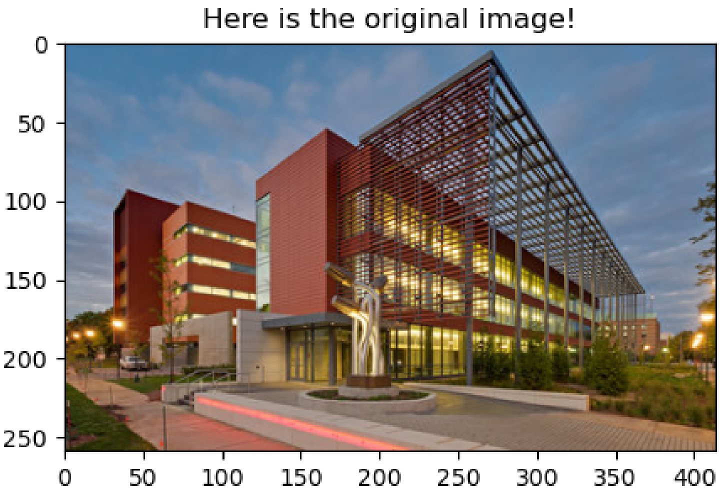
Outline

Outline of today's lecture

1. HW3 and MP3
2. Local averaging
3. Convolution
4. Differencing
5. Edge Detection

Outline

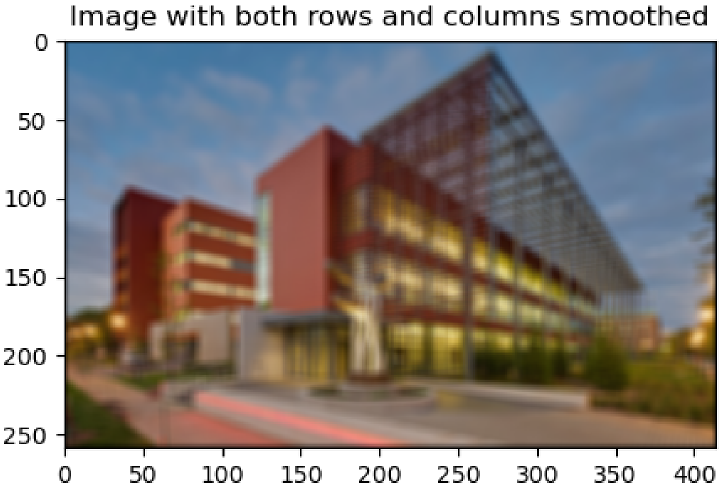
How do you treat an image as a signal?



How do you treat an image as a signal?

- An RGB image is a signal in three dimensions: $f[i,j,k] =$ intensity of the signal in the i^{th} row, j^{th} column, and k^{th} color.
- $f[i,j,k]$, for each (i,j,k), is either stored as an integer or a floating point number:
  - Floating point: usually $x \in [0,1]$, so $x=0$ means dark, $x=1$ means bright.
  - Integer: usually $x \in \{0,\ldots,255\}$, so $x=0$ means dark, $x=255$ means bright.
- The three color planes are usually:
  - $k=0$: Red
  - $k=1$: Blue
  - $k=2$: Green

Local averaging



Local averaging

- "Local averaging" means that we create an output image, $y[i,j,k]$, each of whose pixels is an **average** of nearby pixels in $f[i,j,k]$.
- For example, if we average along the rows:

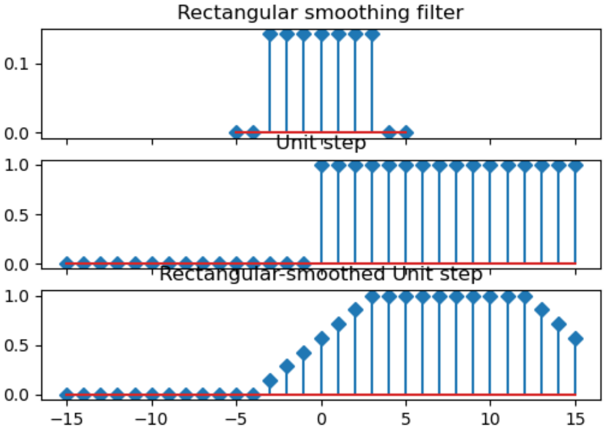
$$y[i,j,k] = \frac{1}{2M+1}\sum_{j'=j-M}^{j+M} f[i,j',k]$$

- If we average along the columns:

$$y[i,j,k] = \frac{1}{2M+1}\sum_{i'=i-M}^{i+M} f[i',j,k]$$

Local averaging of a unit step

The top row are the averaging weights. If it's a 7-sample local average, $(2M+1)=7$, so the averaging weights are each $\frac{1}{2M+1}=\frac{1}{7}$. The middle row shows the input, $f[n]$. The bottom row shows the output, $y[n]$.



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Weighted local averaging

- Suppose we don't want the edges quite so abrupt. We could do that using "weighted local averaging:" each pixel of $y[i,j,k]$ is a **weighted average** of nearby pixels in $f[i,j,k]$, with some averaging weights $g[n]$.
- For example, if we average along the rows:

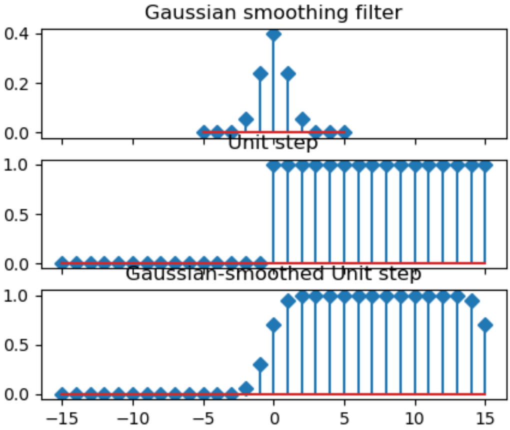
$$y[i,j,k] = \sum_{m=j-M}^{j+M} g[j-m]f[i,m,k]$$

- If we average along the columns:

$$y[i,j,k] = \sum_{i'=i-M}^{i+M} g[i-m]f[m,j,k]$$

Weighted local averaging of a unit step

The top row are the averaging weights, $g[n]$. The middle row shows the input, $f[n]$. The bottom row shows the output, $y[n]$.



Outline

Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

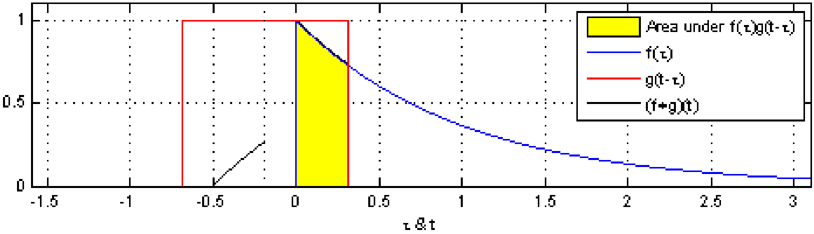
Convolution

$$\boxed{y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]}$$

Here is the pseudocode for convolution:

1. For every output n:
    1. Reverse $g[m]$ in time, to create $g[-m]$.
    2. Shift it to the right by n samples, to create $g[n-m]$.
    3. For every m:
        1. Multiply $f[m]g[n-m]$.
    4. Add them up to create $y[n] = \sum_m g[n-m]f[m]$ for this particular n.
2. Concatenate those samples together, in sequence, to make the signal y.

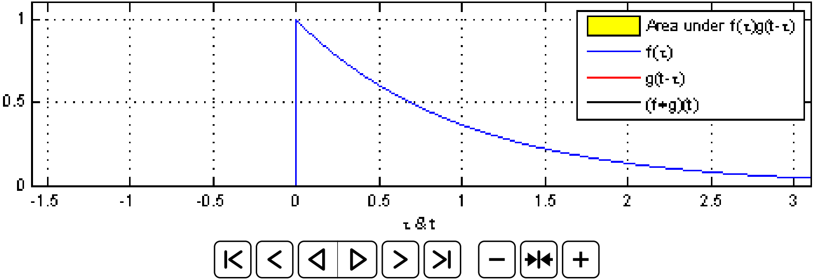
Convolution



by Brian Amberg, CC-SA 3.0,

`https://commons.wikimedia.org/wiki/File:Convolution_of_spiky_function_with_box2.gif`

Convolution



Convolution: how should you implement it?

Answer: use the numpy function, `np.convolve`. In general, if numpy has a function that solves your problem, you are *always* permitted to use it.

numpy.convolve

numpy.**convolve**(*a*, *v*, *mode='full'*) [source]

Returns the discrete, linear convolution of two one-dimensional sequences.

The convolution operator is often seen in signal processing, where it models the effect of a linear time-invariant system on a signal [1]. In probability theory, the sum of two independent random variables is distributed according to the convolution of their individual distributions.

If *v* is longer than *a*, the arrays are swapped before computation.

Parameters:

a : *(N,) array_like*
First one-dimensional input array.

v : *(M,) array_like*
Second one-dimensional input array.

mode : *{'full', 'valid', 'same'}, optional*
'full':

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Differencing is convolution, too

Suppose we want to compute the local difference:

$$y[n] = f[n] - f[n-1]$$

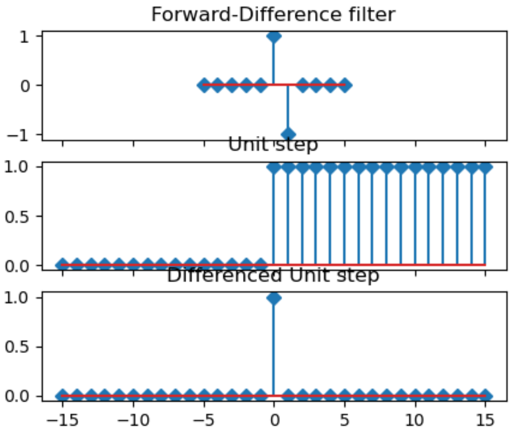
We can do that using a convolution!

$$y[n] = \sum_m f[n-m]h[m]$$

where

$$h[m] = \begin{cases} 1 & m = 0 \\ -1 & m = 1 \\ 0 & \text{otherwise} \end{cases}$$

Differencing as convolution



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Weighted differencing as convolution

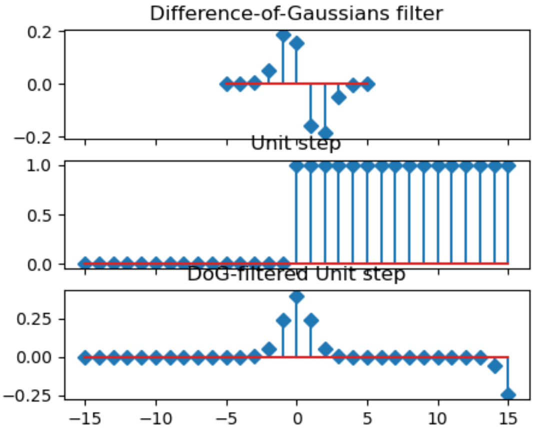
- The formula $y[n] = f[n] - f[n-1]$ is kind of noisy. Any noise in $f[n]$ or $f[n-1]$ means noise in the output.
- We can make it less noisy by
  1. First, compute a weighted average:
     $$y[n] = \sum_m f[m]g[n-m]$$
  2. Then, compute a local difference:
     $$z[n] = y[n] - y[n-1] = \sum_m f[m]\left(g[n-m] - g[n-1-m]\right)$$

  This is exactly the same thing as convolving with

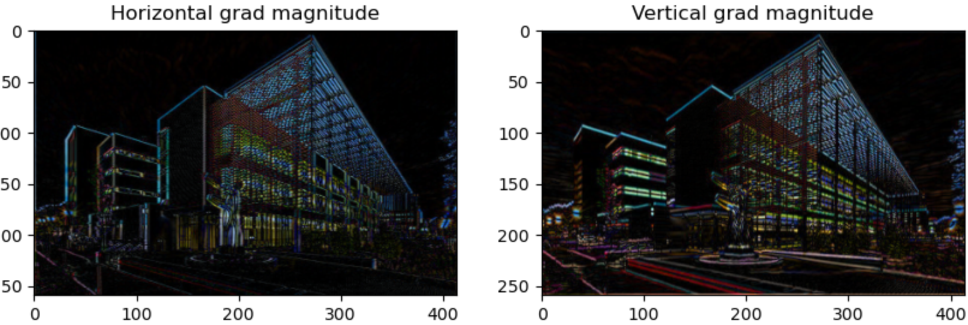
  $$h[n] = g[n] - g[n-1]$$

A difference-of-Gaussians filter

The top row is a "difference of Gaussians" filter, $h[n] = g[n] - g[n-1]$, where $g[n]$ is a Gaussian. The middle row is $f[n]$, the last row is the output $z[n]$.



Difference-of-Gaussians filtering in both rows and columns



Outline

Image gradient

- Suppose we have an image $f[i,j,k]$. The 2D image gradient is defined to be

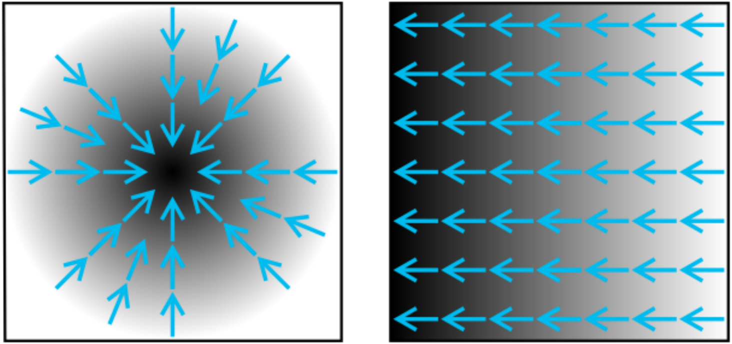
$$\vec{G}[i,j,k] = \left(\frac{df}{di}\right)\hat{i} + \left(\frac{df}{dj}\right)\hat{j}$$

  where $\hat{i}$ is a unit vector in the i direction, $\hat{j}$ is a unit vector in the j direction.
- We can approximate these using the difference-of-Gaussians filter, $h_{dog}[n]$:

$$\frac{df}{di} \approx G_i = h_{dog}[i] * f[i,j,k]$$
$$\frac{df}{dj} \approx G_j = h_{dog}[j] * f[i,j,k]$$

The gradient is a vector

The image gradient, at any given pixel, is a vector. It points in the direction of increasing intensity (this image shows "dark" = greater intensity).



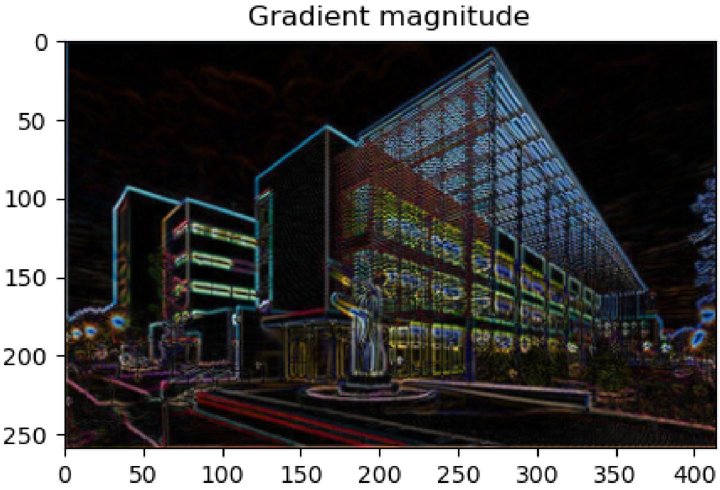
By CWeiske, CC-SA 2.5, https://commons.wikimedia.org/wiki/File:Gradient2.svg

Magnitude of the image gradient

- The image gradient, at any given pixel, is a vector.
- It points in the direction in which intensity is increasing.
- The magnitude of the vector tells you how fast intensity is changing.

$$\|\vec{G}\| = \sqrt{G_i^2 + G_j^2}$$

Magnitude of the gradient = edge detector



Outline

Summary

$$y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$