Lecture 8: Sampling Theorem

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ECE 401: Signal and Image Analysis, Fall 2023

1. Review: Sampling
2. Spectrum Plots
3. Spectrum of Oversampled Signals
4. Spectrum of Undersampled Signals
5. The Sampling Theorem
6. Discrete-Time Fourier Series
7. Summary
8. Written Example

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How to sample a continuous-time signal

Suppose you have some continuous-time signal, $x(t)$, and you'd like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$

Aliasing

- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
  - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

$$f_a = F_s - f$$
$$z_a = z^*$$

    i.e., the sign of all sines will be reversed.
  - If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

$$f_a = f - F_s$$
$$z_a = z$$

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Spectrum Plots

The **spectrum plot** of a periodic signal is a plot with

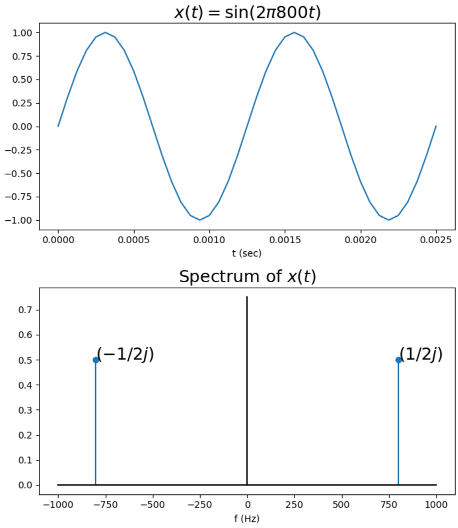
- frequency on the X-axis,
- showing a vertical spike at each frequency component,
- each of which is labeled with the corresponding phasor.

Example: Sine Wave

$$
\begin{aligned}
x(t) &= \sin\left(2\pi 800 t\right) \\
&= \frac{1}{2j}e^{j2\pi 800 t} - \frac{1}{2j}e^{-j2\pi 800 t}
\end{aligned}
$$

The spectrum of $x(t)$ is $\{(-800, -\frac{1}{2j}), (800, \frac{1}{2j})\}$.

Example: Sine Wave

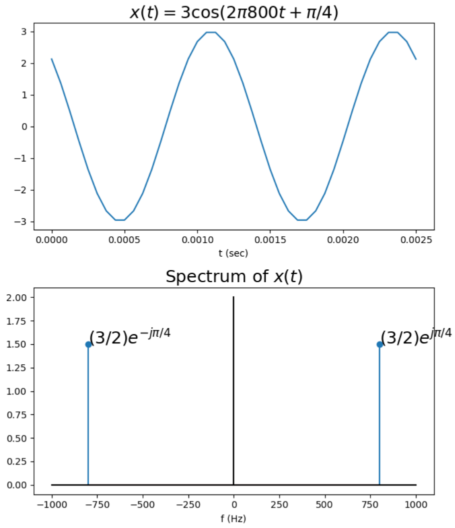


Example: Quadrature Cosine

$$\begin{aligned} x(t) &= 3\cos\left(2\pi 800 t + \frac{\pi}{4}\right) \\ &= \frac{3}{2} e^{j\pi/4} e^{j2\pi 800 t} + \frac{3}{2} e^{-j\pi/4} e^{-j2\pi 800 t} \end{aligned}$$

The spectrum of $x(t)$ is $\{(-800, \frac{3}{2}e^{-j\pi/4}), (800, \frac{3}{2}e^{j\pi/4})\}$.

Example: Quadrature Cosine



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Oversampled Signals

A signal is called **oversampled** if $F_s > 2f$ (e.g., so that sinc interpolation can reconstruct it from its samples).

Spectrum Plot of a Discrete-Time Periodic Signal

The spectrum plot of a **discrete-time periodic signal** is a regular spectrum plot, but with the X-axis relabeled. Instead of frequency in Hertz$=\left[\frac{\text{cycles}}{\text{second}}\right]$, we use

$$\omega\left[\frac{\text{radians}}{\text{sample}}\right]=\frac{2\pi\left[\frac{\text{radians}}{\text{cycle}}\right]f\left[\frac{\text{cycles}}{\text{second}}\right]}{F_s\left[\frac{\text{samples}}{\text{second}}\right]}$$

How do we plot the aliasing?

Remember that a discrete-time signal has energy at

- f and $-f$, but also $F_s - f$ and $-F_s + f$, and $F_s + f$ and $-F_s - f$, and...
- ω and $-\omega$, but also $2\pi - \omega$ and $-2\pi + \omega$, and $2\pi + \omega$ and $-2\pi - \omega$, and...

Which ones should we plot? Answer: **plot all of them!** Usually we plot a few nearest the center, then add "..." at either end, to show that the plot continues forever.

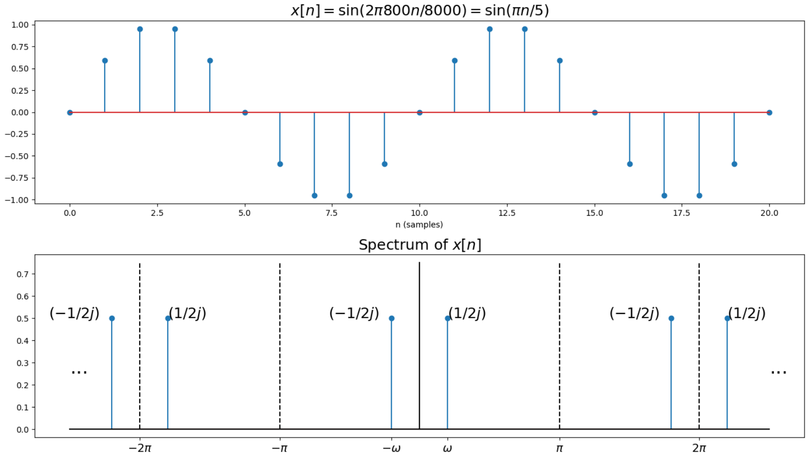
Example: Sine Wave

Let's sample at $F_s = 8000$ samples/second.

$$\begin{aligned} x[n] &= \sin\left(2\pi 800 n/8000\right) \\ &= \sin\left(\pi n/5\right) \\ &= \frac{1}{2j} e^{j\pi n/5} - \frac{1}{2j} e^{-j\pi n/5} \end{aligned}$$

The spectrum of $x[n]$ is $\{\ldots, (-\pi/5, -\frac{1}{2j}), (\pi/5, \frac{1}{2j}), \ldots\}$.

Example: Sine Wave

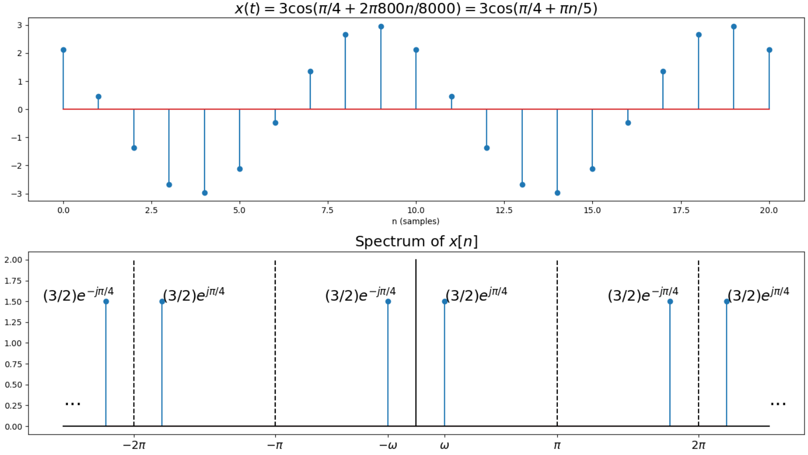


Example: Quadrature Cosine

$$\begin{aligned} x[n] &= 3\cos\left(2\pi 800n/8000 + \frac{\pi}{4}\right) \\ &= 3\cos\left(\pi n/5 + \frac{\pi}{4}\right) \\ &= \frac{3}{2}e^{j\pi/4}e^{j\pi n/5} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi n/5} \end{aligned}$$

The spectrum of $x[n]$ is $\{\ldots, (-\pi/5, \frac{3}{2}e^{-j\pi/4}), (\pi/5, \frac{3}{2}e^{j\pi/4}), \ldots\}$.

Example: Quadrature Cosine



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Undersampled Signals

A signal is called **undersampled** if $F_s < 2f$ (e.g., so that sinc interpolation can't reconstruct it from its samples).

. . . but Aliasing?

Remember that a discrete-time signal has energy at

- f and $-f$, but also $F_s - f$ and $-F_s + f$, and $F_s + f$ and $-F_s - f$, and. . .
- ω and $-\omega$, but also $2\pi - \omega$ and $-2\pi + \omega$, and $2\pi + \omega$ and $-2\pi - \omega$, and. . .

We still want to plot all of these, but now ω and $-\omega$ won't be the spikes closest to the center. Instead, some other spike will be closest to the center.

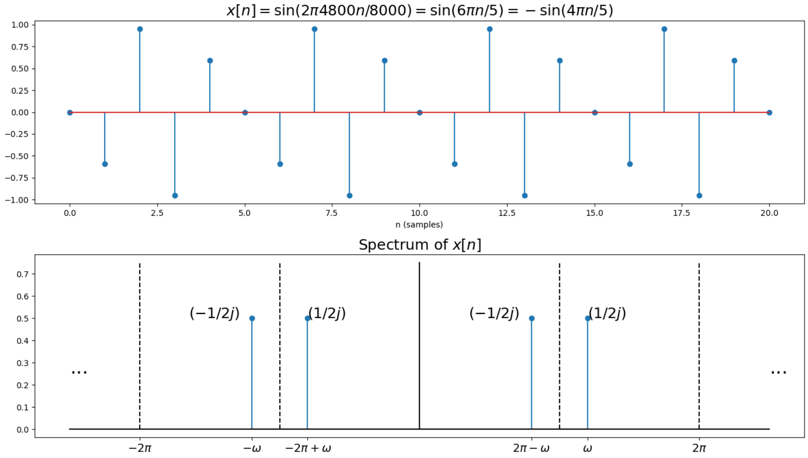
Example: Sine Wave

Let's still sample at $F_s = 8000$, but we'll use a sine wave at $f = 4800$Hz, so it gets undersampled.

$$\begin{aligned} x[n] &= \sin\left(2\pi 4800 n/8000\right) \\ &= \sin\left(6\pi n/5\right) \\ &= -\sin\left(4\pi n/5\right) \\ &= -\frac{1}{2j} e^{j4\pi n/5} + \frac{1}{2j} e^{j4\pi n/5} \end{aligned}$$

The spectrum of $x[n]$ is $\{\ldots, (-4\pi/5, \frac{1}{2j}), (4\pi/5, -\frac{1}{2j}), \ldots\}$.

Example: Sine Wave

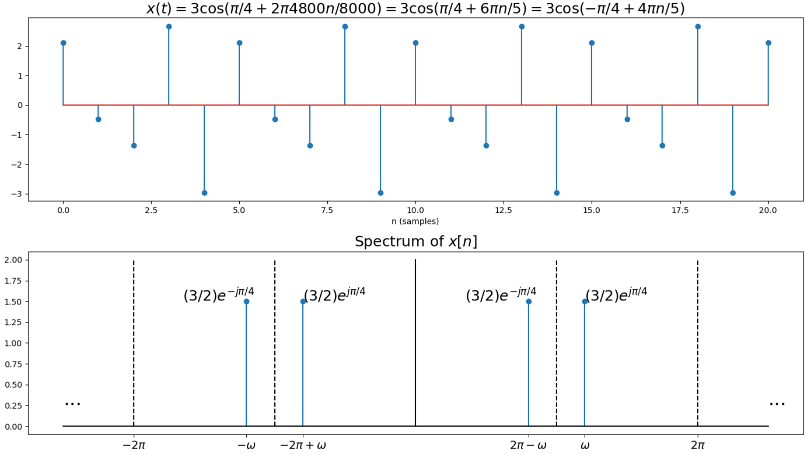


Example: Quadrature Cosine

$$
\begin{aligned}
x[n] &= 3\cos\left(2\pi 4800n/8000 + \frac{\pi}{4}\right)\\
&= 3\cos\left(6\pi n/5 + \frac{\pi}{4}\right)\\
&= 3\cos\left(4\pi n/5 - \frac{\pi}{4}\right)\\
&= \frac{3}{2}e^{-j\pi/4}e^{j4\pi n/5} + \frac{3}{2}e^{j\pi/4}e^{-j4\pi n/5}
\end{aligned}
$$

The spectrum of $x[n]$ is
$\{\ldots, (-4\pi/5, \frac{3}{2}e^{j\pi/4}), (4\pi/5, \frac{3}{2}e^{-j\pi/4}), \ldots\}$.

Example: Quadrature Cosine



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General periodic continuous-time signals

Let's assume that $x(t)$ is periodic with some period T_0, therefore it has a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} = \sum_{k=0}^{\infty} 2|X_k| \cos\left(\frac{2\pi kt}{T_0} + \angle X_k\right)$$

Eliminate the aliased tones

We already know that $e^{j2\pi kt/T_0}$ will be aliased if $|k|/T_0 > F_N$. So let's assume that the signal is **band-limited:** it contains no frequency components with frequencies larger than $F_S/2$.
That means that the only X_k with nonzero energy are the ones in the range $-\frac{N-1}{2} \le k \le \frac{N-1}{2}$, where $\frac{N-1}{2T_0} < \frac{F_s}{2}$:

$$x(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kt/T_0}$$

Notice that, counting the $k = 0$ term, there are an odd number of harmonics (N is odd), in the range $-\frac{N-1}{2} \le k \le \frac{N-1}{2}$.

Sample that signal!

Now let's sample that signal, at sampling frequency F_S:

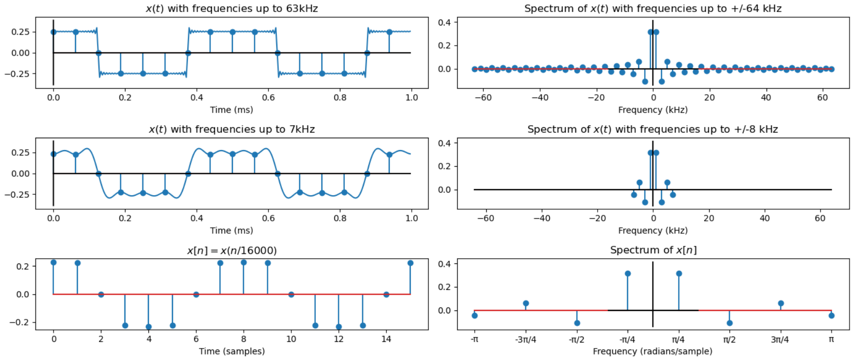
$$x[n] = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kn/F_S T_0}$$

$$= \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{jk\omega_0 n},$$

where the discrete-time fundamental frequency, expressed in radians/sample, is

$$\omega_0 = \frac{2\pi F_0}{F_s} = \frac{2\pi}{F_s T_0}$$

Spectrum of a sampled periodic signal



The sampling theorem

As long as $-\pi \leq \omega_k \leq \pi$, we can recreate the continuous-time signal by either (1) using sinc interpolation, or (2) regenerating a continuous-time signal with the corresponding frequency:

$$f_k \left[\frac{\text{cycles}}{\text{second}}\right] = \frac{\omega_k \left[\frac{\text{radians}}{\text{sample}}\right] \times F_S \left[\frac{\text{samples}}{\text{second}}\right]}{2\pi \left[\frac{\text{radians}}{\text{cycle}}\right]}$$

$$x[n] = \cos(\omega_k n + \theta_k) \quad \rightarrow \quad x(t) = \cos(2\pi f_k t + \theta_k)$$

The sampling theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_S)$ if the samples are taken at a rate $F_s = 1/T_s$ that is $F_S \geq 2f_{max}$.

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Continuous-Time Fourier Series

Suppose we have a continuous-time periodic signal that is already band-limited, so its highest frequency is $\frac{N-1}{2T_0} < \frac{F_s}{2}$. Its continuous-time Fourier series is

$$x(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kt/T_0}$$

Is it periodic in discrete time?

If the period T_0 is an integer number of samples ($T_0 = N/F_s$), then this signal is also periodic in discrete time:

$$x(t) = x(t + T_0)$$
$$x[n] = x[n + N]$$

Discrete-Time Fourier Series

If the signal is periodic in discrete time, then, by sampling its continuous-time Fourier series, we get its **discrete-time Fourier series**:

$$\begin{aligned} x[n] &= \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kn/F_S T_0} \\ &= \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kn/N}, \qquad = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{jk\omega_0 n}, \end{aligned}$$

where the discrete-time fundamental frequency, expressed in radians/sample, is

$$\omega_0 = \frac{2\pi F_0}{F_s} = \frac{2\pi}{F_s T_0} = \frac{2\pi}{N}$$

DTFS Coefficients

Remember that the Fourier series coefficients are computed as

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

If the signal is periodic in discrete time (if T_0 is an integer number of samples), then we can compute exactly the same coefficients by averaging in discrete time:

$$X_k = \frac{1}{N}\sum_0^{N-1} x[n]e^{-j2\pi kn/N}$$

Discrete-Time Fourier Series

If $x[n]$ is periodic with period N, then it has a Fourier series

$$x[n] = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j\frac{2\pi kn}{N}},$$

whose coefficients can be computed as

$$X_k = \frac{1}{N}\sum_{0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

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Written Example

Let $x(t)$ be a sinusoid with some amplitude, some phase, and some frequency.

- Plot the spectrum of $x(t)$.
- Choose an F_s that undersamples it. Plot the spectrum of $x[n]$.