Lecture 6: Sampling and Aliasing

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2023

Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Fourier's theorem

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0)=x(t)$$

can be written as

$$x(t)=\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k=kF_0$, and $a_k=X_k$, and

$$F_0=\frac{1}{T_0}$$

Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Outline

How to sample a continuous-time signal

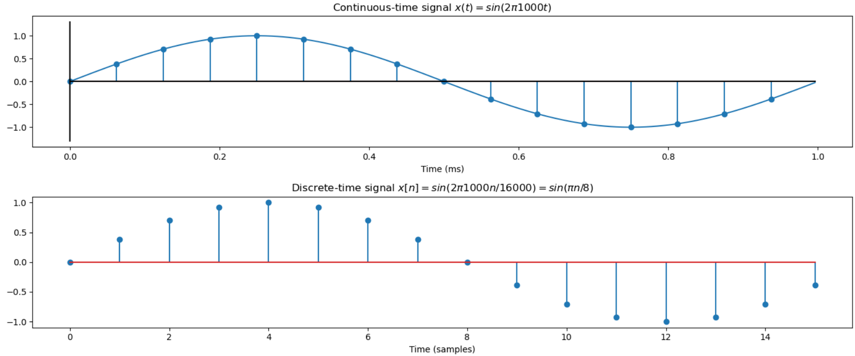
Suppose you have some continuous-time signal, $x(t)$, and you'd like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$

Example: a 1kHz sine wave

For example, suppose $x(t) = \sin(2\pi 1000 t)$. By sampling at $F_s = 16000$ samples/second, we get

$$x[n] = \sin\left(2\pi 1000 \frac{n}{16000}\right) = \sin(\pi n/8)$$



Outline

Can every sine wave be reconstructed from its samples?

The question immediately arises: can every sine wave be reconstructed from its samples?
The answer, unfortunately, is “no.”

Can every sine wave be reconstructed from its samples?

For example, two signals $x_1(t)$ and $x_2(t)$, at 10kHz and 6kHz respectively:

$$x_1(t) = \cos(2\pi 10000 t), \quad x_2(t) = \cos(2\pi 6000 t)$$

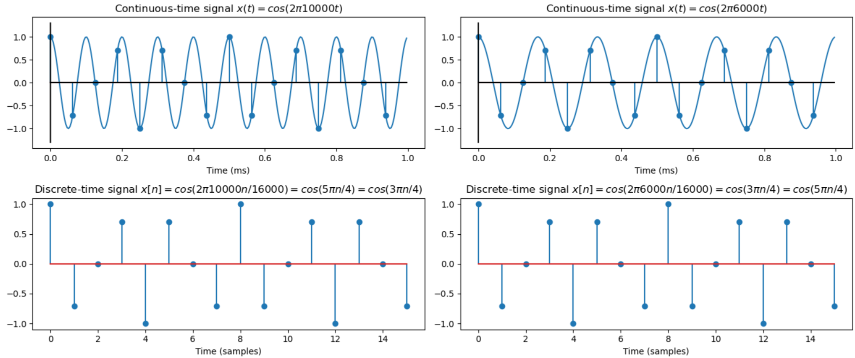
Let's sample them at $F_s = 16,000$ samples/second:

$$x_1[n] = \cos\left(2\pi 10000 \frac{n}{16000}\right), \quad x_2[n] = \cos\left(2\pi 6000 \frac{n}{16000}\right)$$

Simplifying a bit, we discover that $x_1[n] = x_2[n]$. We say that the 10kHz tone has been "aliased" to 6kHz:

$$x_1[n] = \cos\left(\frac{5\pi n}{4}\right) = \cos\left(\frac{3\pi n}{4}\right)$$
$$x_2[n] = \cos\left(\frac{3\pi n}{4}\right) = \cos\left(\frac{5\pi n}{4}\right)$$

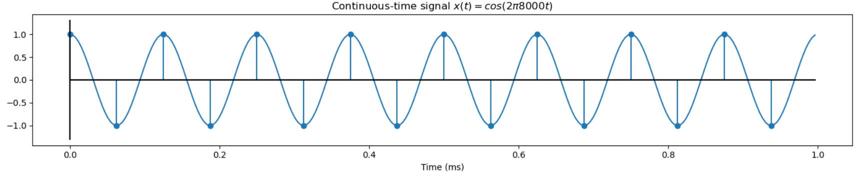
Can every sine wave be reconstructed from its samples?

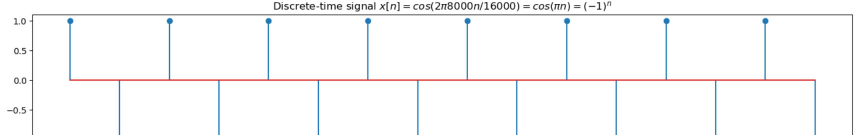


What is the highest frequency that can be reconstructed?

The minimum sampling rate that avoids aliasing is called the **Nyquist rate,** and it is $F_s = 2f$. Conversely, we talk about the the **Nyquist frequency,** $F_N = F_S/2$, which is the highest frequency pure tone that can be reconstructed at sampling rate F_s. If $x(t) = \cos(2\pi F_N t)$, then

$$x[n] = \cos\left(2\pi F_N \frac{n}{F_S}\right) = \cos(\pi n) = (-1)^n$$





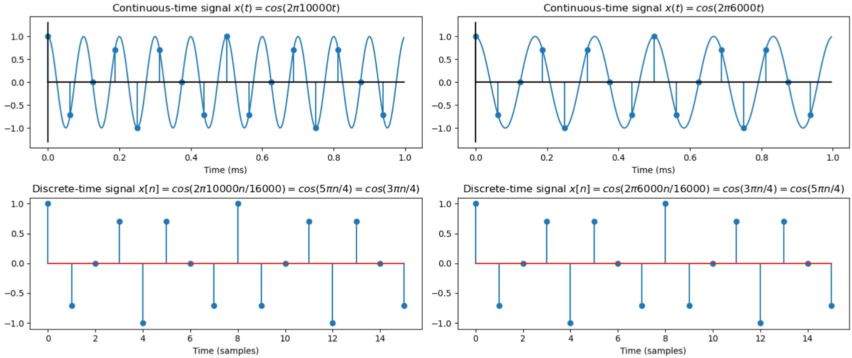
Nyquist rate vs. Nyquist frequency

Unfortunately, due to historical reasons, the terms "Nyquist rate" and "Nyquist frequency" are sort of opposite in meaning:

- The Nyquist rate is the **lowest sampling rate** at which you can sample a signal without aliasing. If the highest frequency in a signal is f, then the Nyquist rate is $F_s = 2f$.
- The Nyquist frequency is the **highest frequency** that will be reproduced without aliasing, i.e., $F_N = F_s/2$.

Sampling below Nyquist rate ⇒ Aliasing to a frequency below the Nyquist frequency

If you try to sample below the **Nyquist rate** ($F_s < 2f$, like the one shown on the left), then the tone gets aliased to a **frequency alias** f_a below the **Nyquist frequency** ($f_a < F_N$, like the one shown on the right).



When does aliasing happen?

Aliasing happens:

- When a continuous-time signal, $x(t) = \cos(2\pi ft)$, is sampled below the Nyquist rate: $F_s < 2f$.
- When a tone has already been sampled at a high enough sampling rate, but then you **downsample** to a rate below Nyquist.

For example, suppose you have sampled at $F_s = 2.88f$, so that you have

$$x[n] = \cos\left(\frac{2\pi f}{F_s} n\right) = \cos\left(\frac{2\pi}{2.88} n\right),$$

but if you then **downsample** by throwing away every second sample,

$$y[n] = x[2n], \quad \text{integer values of } n,$$

then you wind up with a new sampling rate of only $F_s = 1.44f$, which means the signal can be aliased to a lower frequency below Nyquist:

$$y[n] = \cos\left(\frac{2\pi}{1.44} n\right) = \cos\left(\left(2\pi - \frac{2\pi}{1.44}\right) n\right) = \cos\left(\frac{0.88\pi}{1.44} n\right)$$

Outline

Aliased Frequency

Suppose you have a cosine at frequency f:

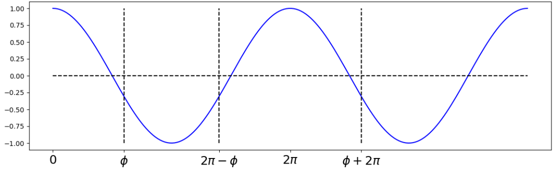
$$x(t) = \cos(2\pi f t)$$

Suppose you sample it at F_s samples/second. If F_s is not high enough, it might get aliased to some other frequency, f_a.

$$x[n] = \cos(2\pi f n/F_s) = \cos(2\pi f_a n/F_s)$$

How can you predict what f_a will be?

Aliased Frequency

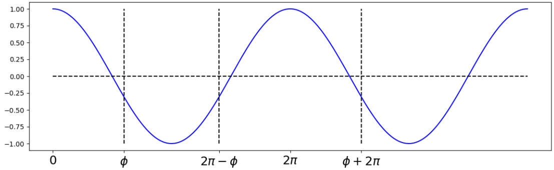


Aliasing comes from two sources:

$$\cos(\phi) = \cos(2\pi n - \phi)$$
$$\cos(\phi) = \cos(\phi - 2\pi n)$$

The equations above are true for any integer n.

Aliased Frequency



Let's plug in $\phi = \frac{2\pi fn}{F_s}$, and $2\pi = \frac{2\pi F_s}{F_s}$. That gives us:

$$\cos\left(\frac{2\pi fn}{F_s}\right) = \cos\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$

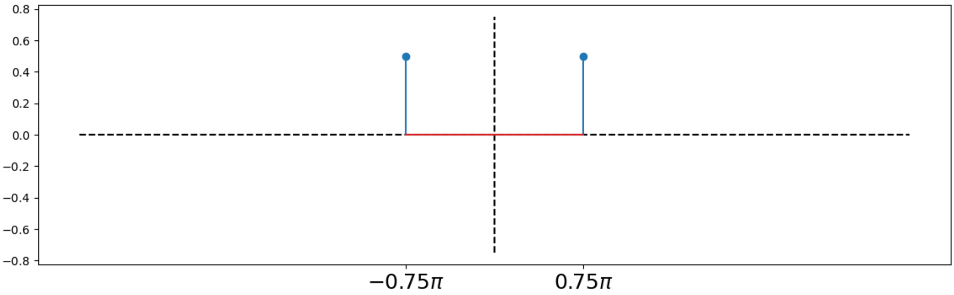
$$\cos\left(\frac{2\pi fn}{F_s}\right) = \cos\left(\frac{2\pi(f - F_s)n}{F_s}\right)$$

So a discrete-time cosine at frequency f is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.

Spectrum of a Continuous-time Cosine

A continuous-time cosine is the sum of two complex exponentials:

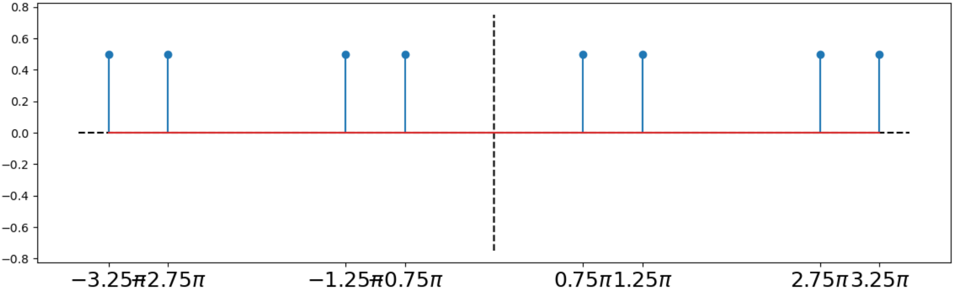
$$\cos(0.75\pi n) = \frac{1}{2}e^{j0.75\pi n} + \frac{1}{2}e^{-j0.75\pi n}$$



Spectrum of a Discrete-time Cosine

A discrete-time cosine is **still** just the sum of two complex exponentials, but each of those two complex exponentials is identical to an exponential at any other multiple of 2π:

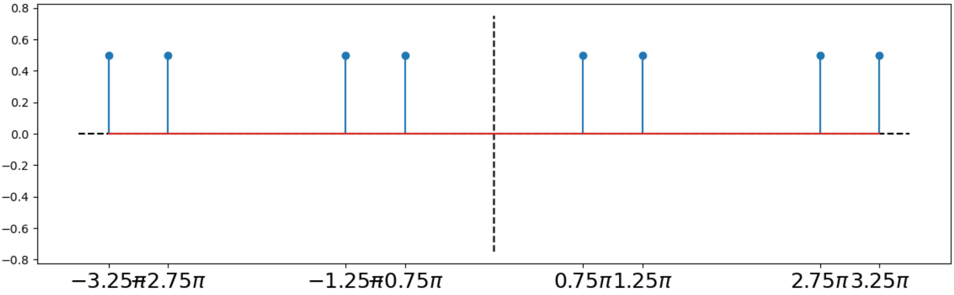
$$e^{j2.75\pi n} = e^{-j2\pi n}e^{j0.75\pi n} = e^{j0.75\pi n}$$



Spectrum of an Aliased Discrete-time Cosine

Now consider what happens as we lower F_s. As we lower F_s, the frequency $\omega = \frac{2\pi f}{F_s}$ gets higher and higher, until aliasing occurs:

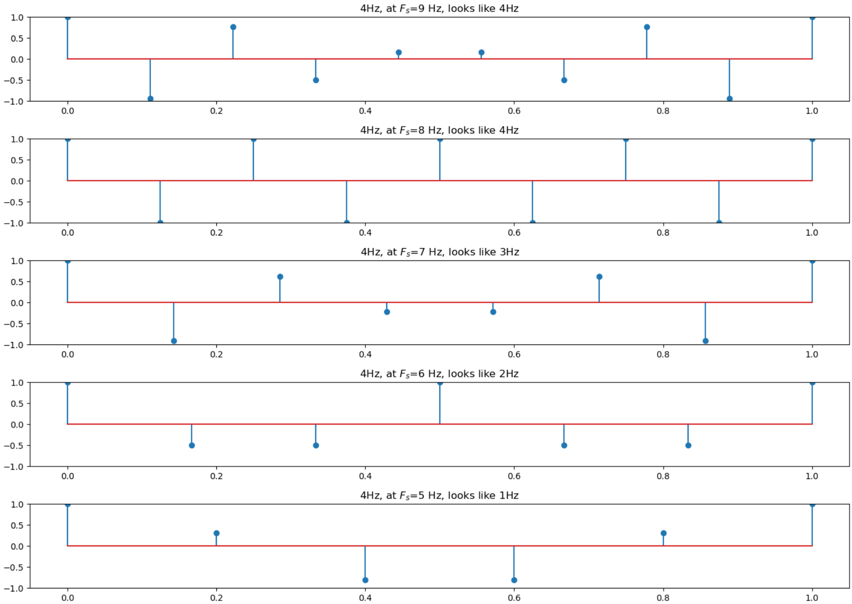
$$\cos(\omega n) = \cos((2\pi - \omega)n)$$



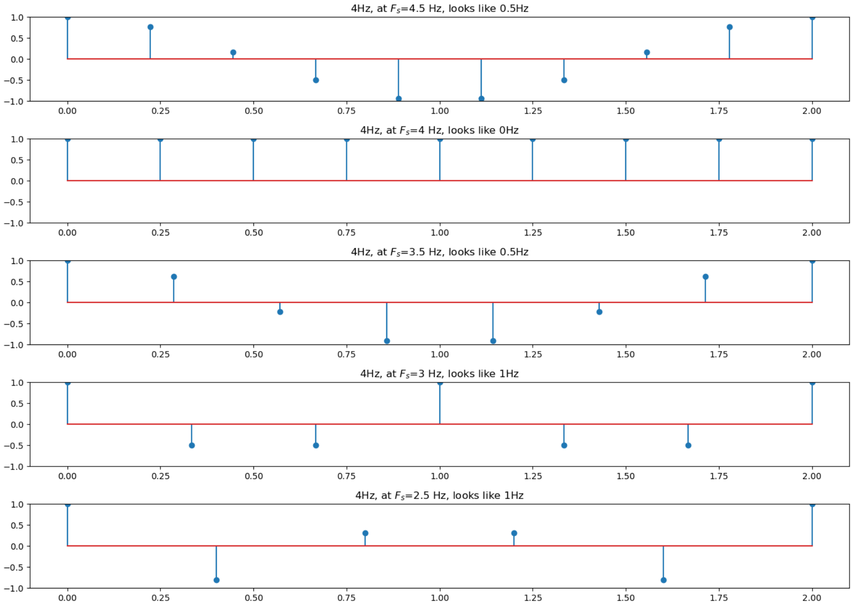
Aliased Frequency

- A discrete-time cosine at frequency f is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.
- So which of those frequencies will we hear when we play the sinusoid back again?
- **ANSWER:** any frequency that can be reconstructed by the analog-to-digital converter. That means any frequency below the Nyquist frequency, $F_N = F_s/2$.

Aliased Frequency



Aliased Frequency



Aliased Frequency

All of the following frequencies are actually **the same frequency** when a cosine is sampled at F_s samples/second.

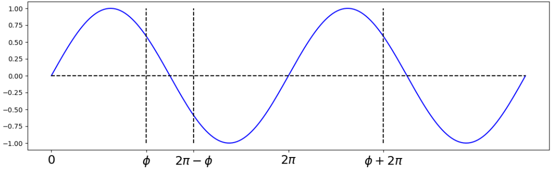
$$f_a \in \{f - \ell F_s, \ell F_s - f : \ell \in \text{any integer}\}$$

The "aliased frequency" is whichever of those is below Nyquist ($F_s/2$). Usually there's only one that's below Nyquist, so you can just look for

$$f_a = \min\left(f - \ell F_s, \ell F_s - f : \ell \in \text{any integer}\right)$$

Outline

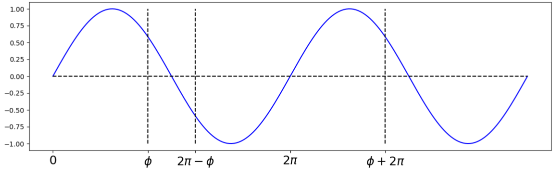
Sine is Different



Sine waves are different for the following reason:

$$\sin(\phi) = -\sin(2\pi n - \phi)$$
$$\sin(\phi) = \sin(\phi - 2\pi n)$$

Sine is Different



Therefore:

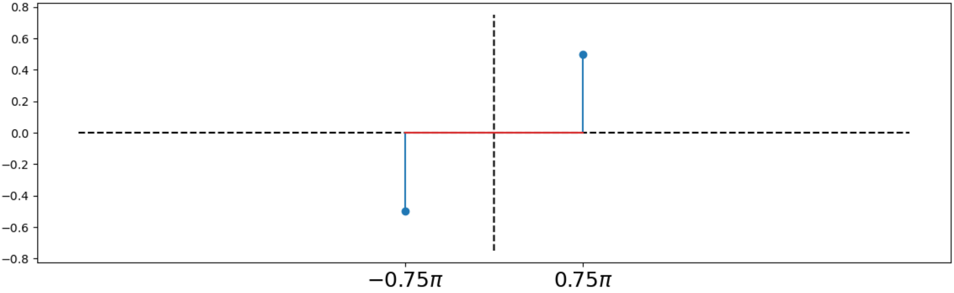
$$\sin\left(\frac{2\pi fn}{F_s}\right) = -\sin\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$
$$\sin\left(\frac{2\pi fn}{F_s}\right) = \sin\left(\frac{2\pi (f - F_s)n}{F_s}\right)$$

So a discrete-time sine at frequency f is also a **negative** sine at frequency $F_s - f$, and a **positive** sine at frequency $f - F_s$.

Spectrum of a Continuous-time Sine

A continuous-time sine is the **difference** of two complex exponentials:

$$\sin(0.75\pi n) = \frac{1}{2j}e^{j0.75\pi n} - \frac{1}{2j}e^{-j0.75\pi n}$$



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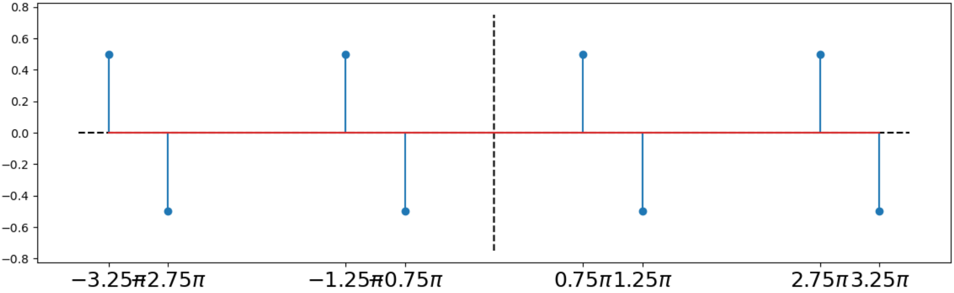

Spectrum of an Aliased Discrete-time Sine

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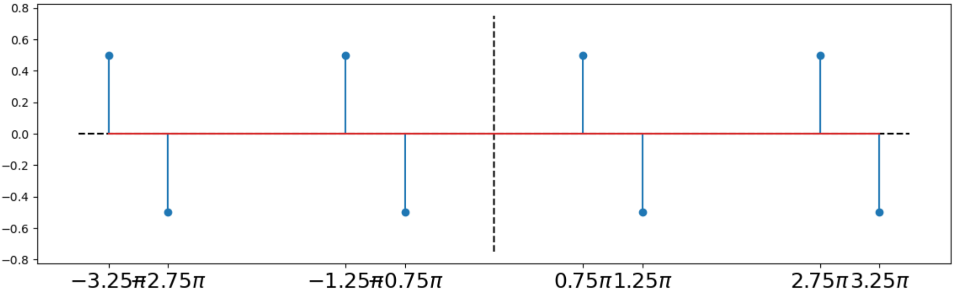


Aliased Phase of a General Phasor

For a general complex exponential, we get:

$$ze^{j\phi} = ze^{j(\phi-2\pi n)} = \left(z^* e^{j(2\pi n-\phi)}\right)^*$$

Therefore:

$$\Re\left\{ze^{j\frac{2\pi fn}{F_s}}\right\} = \Re\left\{ze^{j\frac{2\pi(f-F_s)n}{F_s}}\right\} = \Re\left\{z^* e^{j\frac{2\pi(F_s-f)n}{F_s}}\right\}$$

Aliased Phase of a General Phasor

Suppose we have some frequency f, and we're trying to find its aliased frequency f_a.

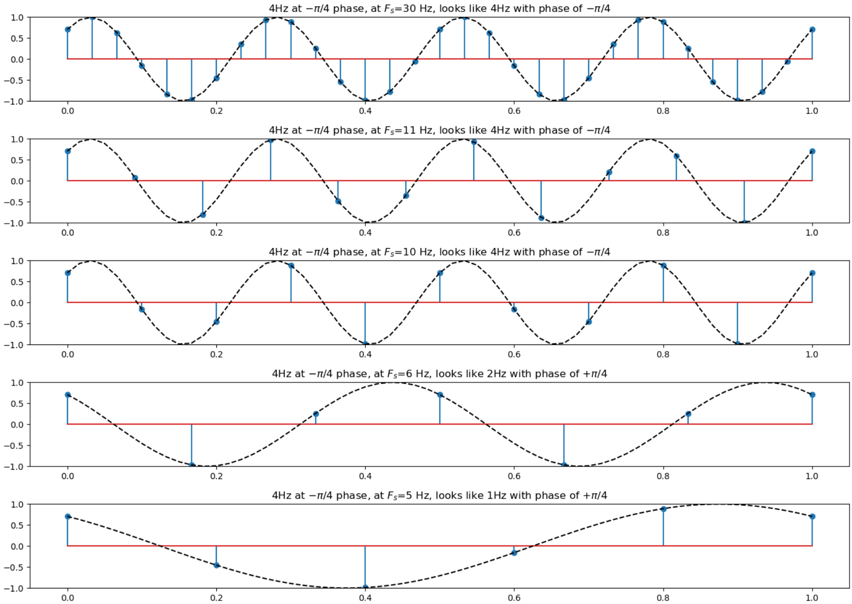
- Among the several possibilities, if $f_a = F_s - f$ is below Nyquist, then that's the frequency we'll hear. Its phasor will be the complex conjugate of the original phasor,

$$z_a = z^*$$

- On the other hand, if $f_a = f - F_s$ is below Nyquist, then that's the frequency we'll hear. Its phasor will be the same as the phasor of the original sinusoid:

$$z_a = z$$

Aliased Phase of a General Phasor



Outline

Summary

- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
  - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

$$f_a = F_s - f$$
$$z_a = z^*$$

    i.e., the sign of all sines will be reversed.
  - If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

$$f_a = f - F_s$$
$$z_a = z$$

Outline

Written Example

Sketch a sinusoid with some arbitrary phase (say, $-\pi/4$). Show where the samples are if it's sampled:

- more than twice per period
- more than once per period, but less than twice per period
- less than once per period