Lecture 5: Operations on Periodic Signals

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Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Properties of the Spectrum

- **Scaling:**

$$y(t) = Gx(t) \Leftrightarrow a_k \rightarrow Ga_k$$

- **Add a Constant:**

$$y(t) = x(t) + C \Leftrightarrow a_k \rightarrow \begin{cases} a_0 + C & k = 0 \\ a_k & \text{otherwise} \end{cases}$$

- **Add Signals:** Suppose that the n^{th} frequency of $x(t)$, m^{th} frequency of $y(t)$, and k^{th} frequency of $z(t)$ are all the same frequency. Then

$$z(t) = x(t) + y(t) \Leftrightarrow a_k \rightarrow a_n + a_m$$

Properties of the Spectrum

- **Time Shift:** Shifting to the right, in time, by τ seconds:

$$y(t) = x(t-\tau) \Leftrightarrow a_k \rightarrow a_k e^{-j2\pi f_k \tau}$$

- **Frequency Shift:** Shifting upward in frequency by F Hertz:

$$y(t) = x(t)e^{j2\pi Ft} \Leftrightarrow f_k \rightarrow f_k + F$$

- **Differentiation:**

$$y(t) = \frac{dx}{dt} \Leftrightarrow a_k \rightarrow j2\pi f_k a_k$$

Outline

Fourier Analysis and Synthesis

- **Fourier Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt$$

- **Fourier Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Fourier Analysis and Synthesis

Compare this:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

to this:

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

we see that a Fourier series is a spectrum in which the k^{th} frequency is $f_k = kF_0$, and the k^{th} phasor is $a_k = X_k$.

Scaling Property for Fourier Series

The scaling property for spectra is

$$y(t) = Gx(t) \Leftrightarrow a_k \rightarrow Ga_k$$

Plugging in $f_k = kF_0$ and $a_k = X_k$, we get

$$y(t) = Gx(t) \Leftrightarrow Y_k = GX_k$$

Constant Offset Property

$$y(t) = x(t) + C \Leftrightarrow a_k \rightarrow \begin{cases} a_0 + C & k = 0 \\ a_k & \text{otherwise} \end{cases}$$

Plugging in $f_k = kF_0$ and $a_k = X_k$, we get

$$y(t) = x(t) + C \Leftrightarrow Y_k = \begin{cases} X_0 + C & k = 0 \\ X_k & \text{otherwise} \end{cases}$$

Signal Addition Property

Suppose that $x(t)$ and $y(t)$ have the same fundamental frequency, F_0. In that case they have the same frequencies $f_k = kF_0$ in their spectra, so

$$z(t) = x(t) + y(t) \Leftrightarrow Z_k = X_k + Y_k$$

Time Shift Property for Fourier Series

The **Time Shift** property of a spectrum is that, if you shift the signal $x(t)$ to the right by τ seconds, then:

$$y(t) = x(t-\tau) \Leftrightarrow a_k \rightarrow a_k e^{-j2\pi f_k \tau}$$

Plugging in $f_k = kF_0$ and $a_k = X_k$, we get

$$y(t) = x(t-\tau) \Leftrightarrow Y_k = X_k e^{-j2\pi k F_0 \tau}$$

Frequency Shift Property for Fourier Series

The **Frequency Shift** property for spectra says that if we multiply by a complex exponential in the time domain, that shifts the entire spectrum by F:

$$y(t) = x(t)e^{j2\pi Ft} \Leftrightarrow f_k \rightarrow f_k + F$$

Suppose that the shift frequency is $F = dF_0$, i.e., it's some integer, d, times the fundamental. Then

- The phasor X_k is no longer active at kF_0; instead, now it's active at $(k+d)F_0$
- We can say that Y_k, the phasor at frequency kF_0, is the same as X_{k-d}, the phase at frequency $(k-d)F_0$.

$$y(t) = x(t)e^{j2\pi Ft} \Leftrightarrow Y_k = X_{k-d}$$

Differentiation Property for Fourier Series

The **Differentiation** property for spectra is

$$y(t) = \frac{dx}{dt} \Leftrightarrow a_k \rightarrow j2\pi f_k a_k$$

If we plug in $f_k = kF_0$ and $a_k = X_k$, we get

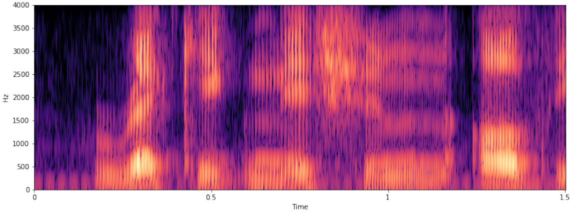
$$y(t) = \frac{dx}{dt} \Leftrightarrow Y_k = j2\pi k F_0 X_k$$

So differentiation in the time domain means multiplying by k in the frequency domain.

Outline

Spectrogram

Many signals have time-varying fundamental frequencies. We show their spectral content by doing Fourier analysis once per 10ms or so, and plotting the log-magnitude Fourier series coefficients as an image called a **spectrogram.** For example, the spectrogram below is from part of Nicholas James Bridgewater's reading of the Universal Declaration of Human Rights



Spectrogram

The textbook demo page on spectrograms has more good examples.

Chirp

You might not have noticed this, but we can write a pure tone as

$$x(t) = e^{j\phi}$$

where $\phi = 2\pi ft$ is the instantaneous phase. Notice that the relationship between frequency and phase is

$$f = \frac{1}{2\pi}\frac{d\phi}{dt}$$

Chirp

In the same way, we can usefully describe the **instantaneous frequency** of a chirp. For example, consider the linear chirp signal:

$$x(t) = e^{jat^2}$$

In this case, $\phi = at^2$, so

$$f = \frac{1}{2\pi}\frac{d\phi}{dt} = \frac{a}{2\pi}t$$

The frequency is now changing as a function of time. Please look at the textbook demo page for more cool examples.

Outline

Handwritten Examples

At this point, let's try doing some handwritten examples.

- Try picking any fundamental frequency. Pick any two harmonics of the fundamental. Put sine waves at both of those harmonics, and add them together to form $x(t)$. What are the resulting Fourier series coefficients? Try sketching this signal.
- Try using the differentiation property to find out what happens when $y(t) = dx/dt$.
- Try using the scaling or constant-shift property.

Outline

Spectral Properties of Fourier Series

- **Scaling:**

$$y(t) = Gx(t) \Leftrightarrow Y_k = GX_k$$

- **Add a Constant:**

$$y(t) = x(t) + C \Leftrightarrow Y_k = \begin{cases} X_0 + C & k = 0 \\ X_k & \text{otherwise} \end{cases}$$

- **Add Signals:** Suppose that $x(t)$ and $y(t)$ have the same fundamental frequency, then

$$z(t) = x(t) + y(t) \Leftrightarrow Z_k = X_k + Y_k$$

Spectral Properties of Fourier Series

- **Time Shift:** Shifting to the right, in time, by τ seconds:

$$y(t) = x(t-\tau) \Leftrightarrow Y_k = a_k e^{-j2\pi k F_0 \tau}$$

- **Frequency Shift:** Shifting upward in frequency by F Hertz:

$$y(t) = x(t) e^{j2\pi d F_0 t} \Leftrightarrow Y_k = X_{k-d}$$

- **Differentiation:**

$$y(t) = \frac{dx}{dt} \Leftrightarrow Y_k = j2\pi k F_0 X_k$$