Lecture 4: Fourier Series

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ECE 401: Signal and Image Analysis, Fall 2023

Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\,(x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Fourier's theorem

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

Analysis and Synthesis

- **Fourier Synthesis** is the process of generating the signal, $x(t)$, given its spectrum. Last lecture, you learned how to do this, in general.
- **Fourier Analysis** is the process of finding the spectrum, X_k, given the signal $x(t)$. I'll tell you how to do that today.

Outline

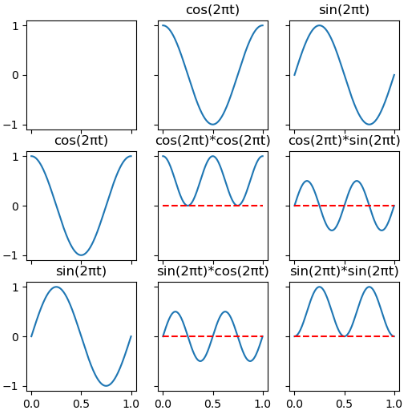
Orthogonality

Two functions $f(t)$ and $g(t)$ are said to be **orthogonal**, over some period of time T, if

$$\int_0^T f(t)g(t) = 0$$

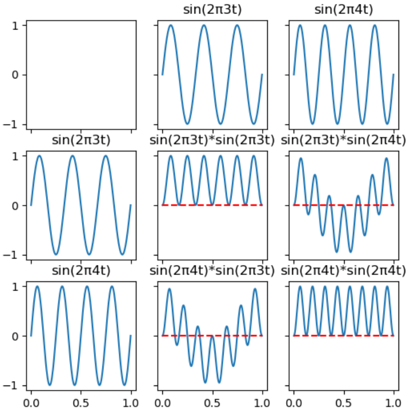
Sine and Cosine are Orthogonal

For example, $\sin(2\pi t)$ and $\cos(2\pi t)$ are orthogonal over the period $0 \leq t \leq 1$:



Sinusoids at Different Frequencies are Orthogonal

Similarly, sinusoids at different frequencies are orthogonal over any time segment that contains an integer number of periods:



How to use orthogonality

Suppose we have a signal that is known to be

$$x(t) = a\cos(2\pi 3t) + b\sin(2\pi 3t) + c\cos(2\pi 4t) + d\sin(2\pi 4t) + \ldots$$

...but we don't know a, b, c, d, etc. Let's use orthogonality to figure out the value of b:

$$\begin{aligned}\int_0^1 x(t)\sin(2\pi 3t)dt = a&\int_0^1 \cos(2\pi 3t)\sin(2\pi 3t)dt\\ + b&\int_0^1 \sin(2\pi 3t)\sin(2\pi 3t)dt\\ + c&\int_0^1 \cos(2\pi 4t)\sin(2\pi 3t)dt\\ + e&\int_0^1 \sin(2\pi 4t)\sin(2\pi 3t)dt + \ldots\end{aligned}$$

How to use orthogonality

. . . which simplifies to

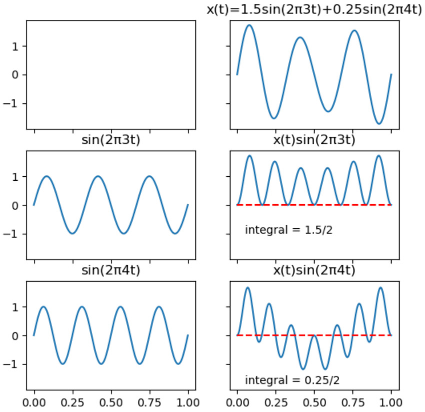
$$\int_0^1 x(t)\sin(2\pi 3t)dt = 0 + b\int_0^1 \sin^2(2\pi 3t)dt + 0 + 0 + \dots$$

The average value of $\sin^2(t)$ is $1/2$, so

$$\int_0^1 x(t)\sin(2\pi 3t)dt = \frac{b}{2}$$

If we **don't** know the value of b, but we **do** know how to integrate $\int x(t)\sin(2\pi 3t)dt$, then we can find the value of b from the formula above.

How to use orthogonality



How to use Orthogonality: Fourier Series

We still have one problem. Integrating $\int x(t)\cos(2\pi 4t)dt$ is hard—lots of ugly integration by parts and so on. What do we do?

1. **Fourier Series:** Instead of cosine, use complex exponential:

$$\int x(t)e^{-j2\pi ft}dt$$

That integral is still hard, but it's always easier than $\int x(t)\cos(2\pi 4t)dt$. It can usually be solved with some combination of integration by parts, variable substitution, etc.

Outline

Fourier's Theorem

Remember Fourier's theorem. He said that any periodic signal, with a period of T_0 seconds, can be written

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

What I want to do, now, is to prove that if you know $x(t)$, you can find its Fourier series coefficients using the following formula:

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

Fourier's Theorem

Remember Fourier's theorem. He said that any periodic signal, with a period of T_0 seconds, can be written

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

I'm going to find the formula for X_k using the following idea:

- **Orthogonality:** $e^{-j2\pi \ell t/T_0}$ is orthogonal to $e^{j2\pi kt/T_0}$ for any $\ell \neq k$.

Fourier's Theorem and Orthogonality

Orthogonality: start with $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$, and multiply both sides by $e^{-j2\pi\ell t/T_0}$:

$$x(t)e^{-2\pi\ell t/T_0} = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi(k-\ell)t/T_0}$$

Now integrate both sides of that equation, over any complete period:

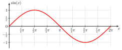
$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi\ell t/T_0} dt = \sum_{k=-\infty}^{\infty} X_k \frac{1}{T_0}\int_0^{T_0} e^{j2\pi(k-\ell)t/T_0} dt$$

Fourier's Theorem and Orthogonality

Now think really hard about what's inside that integral sign:

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(k-\ell)t/T_0}dt$$
$$= \frac{1}{T_0}\int_0^{T_0}\cos\left(\frac{2\pi(k-\ell)t}{T_0}\right)dt$$
$$+j\frac{1}{T_0}\int_0^{T_0}\sin\left(\frac{2\pi(k-\ell)t}{T_0}\right)dt$$

- If $k\neq\ell$, then we're integrating a cosine and a sine over $k-\ell$ periods. That integral is always zero.



- If $k=\ell$, then we're integrating

$$\frac{1}{T_0}\int_0^{T_0}\cos(0)dt + j\frac{1}{T_0}\int_0^{T_0}\sin(0)dt = 1$$

Fourier Series: Analysis

So, because of orthogonality:

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi\ell t/T_0}dt = \sum_{k=-\infty}^{\infty} X_k \frac{1}{T_0}\int_0^{T_0} e^{j2\pi(k-\ell)t/T_0}dt$$

$$= \ldots + 0 + 0 + 0 + X_\ell + 0 + 0 + 0 + \ldots$$

Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

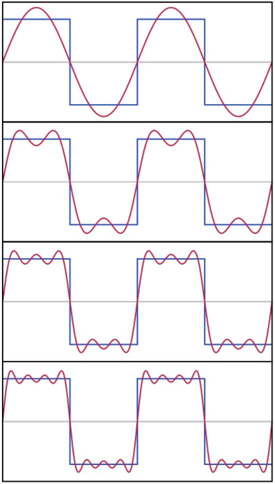
$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

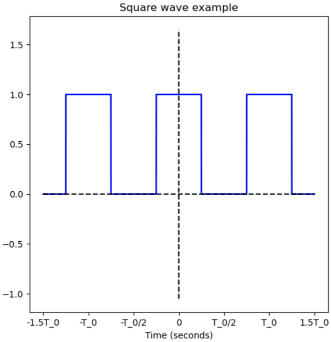
Outline

Fourier series: Square wave example



Square wave example

Let's use a square wave with a nonzero DC value, like this one:

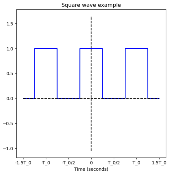


$$x(t) = \begin{cases} 1 & -T_0/4 < t < T_0/4 \\ 0 & \text{otherwise} \end{cases}$$

Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

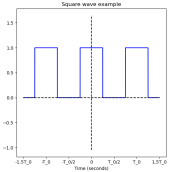
$$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi kt/T_0}dt$$



Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi kt/T_0}dt$$
$$= \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0}dt$$

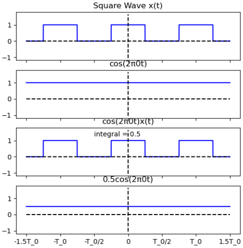


Square wave: the X_0 term

$$X_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0}dt$$

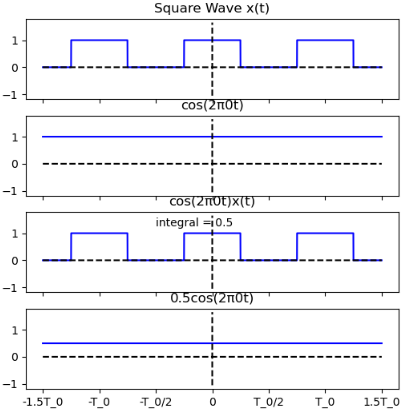
...but if $k=0$, then $e^{-j2\pi kt/T_0}=1$!!!

$$X_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} 1dt = \frac{1}{2}$$



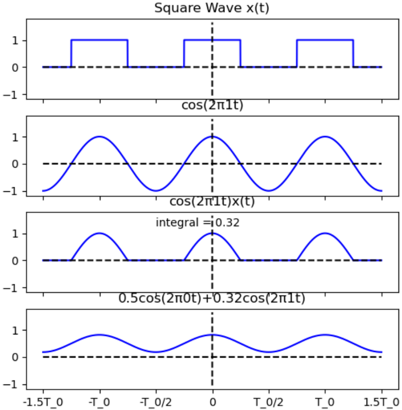
Square wave: the X_0 term

$$X_0 = \frac{1}{2}$$



Square wave: the X_k terms, $k \neq 0$

$$X_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0}dt$$

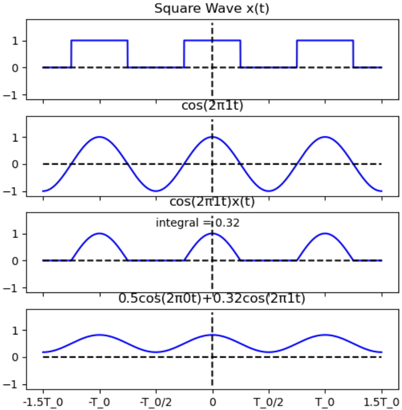


Square wave: the X_k terms, $k \neq 0$

$$
\begin{aligned}
X_k &= \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0}dt \\
&= \frac{1}{T_0}\left(\frac{1}{-j2\pi k/T_0}\right)\left[e^{-j2\pi kt/T_0}\right]_{-T_0/4}^{T_0/4} \\
&= \left(\frac{j}{2\pi k}\right)\left[e^{-j2\pi k(T_0/4)/T_0} - e^{-j2\pi k(-T_0/4)/T_0}\right] \\
&= \left(\frac{j}{2\pi k}\right)\left[e^{-j\pi k/2} - e^{j\pi k/2}\right] \\
&= \frac{1}{\pi k}\sin\left(\frac{\pi k}{2}\right) \\
&= \begin{cases} 0 & k \text{ even} \\ \pm\frac{1}{\pi k} & k \text{ odd} \end{cases}
\end{aligned}
$$

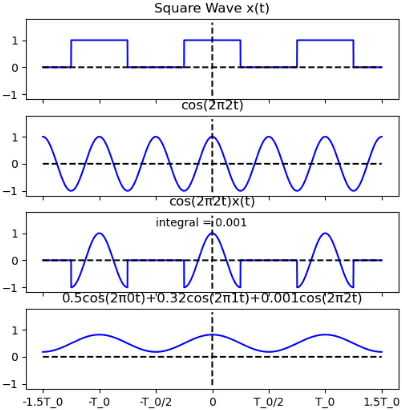
Square wave: the X_1 terms

$$X_1 = \frac{1}{\pi}$$



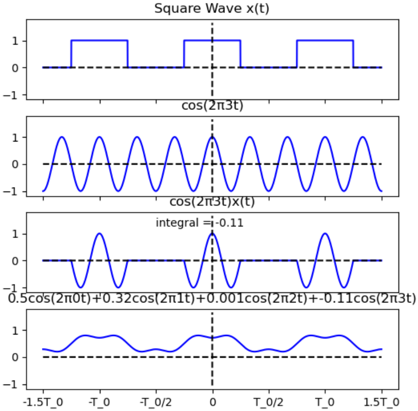
Square wave: the X_2 term

$$X_2 = 0$$



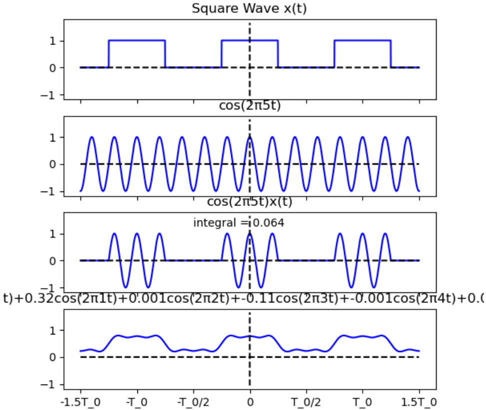
Square wave: the X_3 term

$$X_3 = -\frac{1}{3\pi}$$



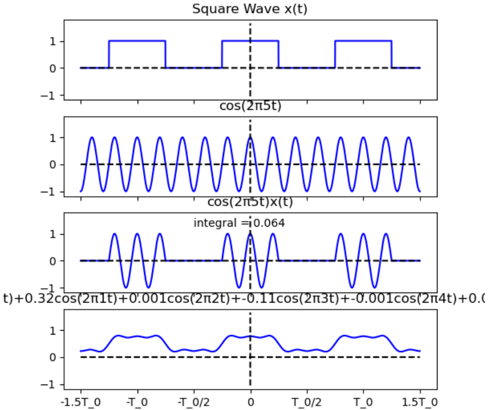
Square wave: the X_5 term

$$X_5 = \frac{1}{5\pi}$$



Square wave: the whole Fourier series

$$x(t) = \frac{1}{2} + \frac{1}{\pi}\left(\cos\left(\frac{2\pi t}{T_0}\right) - \frac{1}{3}\cos\left(\frac{6\pi t}{T_0}\right) + \frac{1}{5}\cos\left(\frac{10\pi t}{T_0}\right) - \frac{1}{7}\ldots\right)$$



Outline

Summary

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$