ECE 401 Signal and Image Analysis
Homework 5

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: 10/30/2023; Due: 11/8/2023
Reading: *DSP First* Chapter 8

Problem 5.1

Consider the signal $x[n] = \delta[n]+\delta[n-2]$. Plot the magnitude DTFT, $|X(\omega)|$, of this signal, for $0 \le \omega < 2\pi$. Draw circles on your plot to show the frequency samples $X[k]$ for a 4-point DFT.

Solution:

$$\begin{aligned} X(\omega) &= 1 + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + e^{-j\omega}) \\ &= e^{-j\omega} 2\cos(\omega) \end{aligned}$$

So

$$|X(\omega)| = 2|\cos(\omega)|$$

The frequency samples corresponding to X_k are at the frequencies $\omega_k = \frac{k\pi}{2}$, and have values of

$$|X[k]| = \begin{cases} 1 & k = 0, 2 \\ 0 & k = 1, 3 \end{cases}$$

Problem 5.2

In this problem, we will repeat Hamming's famous calculation, that resulted in the Hamming window. Consider a slightly modified, even-symmetric raised-cosine window,

$$w_C[n] = \left((1-a) + a\cos\left(\frac{2\pi n}{N}\right)\right) w_R[n]$$

where a is an arbitrary constant, whose value has not yet been determined, and $w_R[n]$ is

$$w_R[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

and the total length of the window is $N = 2M + 1$. Recall that the DTFT of an even-symmetric rectangular window is

$$W_R(\omega) = D_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

(a) Use the linearity and frequency-shift properties of the DTFT to find $W_C(\omega)$, the DTFT of $w_C[n]$.

Solution:

$$\begin{aligned} w_C[n] &= \left((1-a) + a\cos\left(\frac{2\pi n}{N}\right)\right) w_R[n] \\ &= (1-a)w_R[n] + \frac{a}{2}e^{j\frac{2\pi n}{N}} w_R[n] + \frac{a}{2}e^{-j\frac{2\pi n}{N}} w_R[n] \end{aligned}$$

Using the linearity and frequency-shift properties of the DTFT, we see that the DTFT is

$$W_C(\omega) = (1-a)D_N(\omega) + \frac{a}{2}D_N\left(\omega - \frac{2\pi}{N}\right) + D_N\left(\omega + \frac{2\pi}{N}\right)$$

(b) Sketch $W_C(\omega)$, for $0 \le \omega \le \frac{10\pi}{N}$. Draw circles at the frequencies that would be sampled by an N-point DFT. Find the values of $W_C[k]$ for all k in the range $0 \le k \le N-1$, as functions of a and N.

Solution: The DTFT should look like a Dirichlet function, but with its main lobe twice as wide as a normal Dirichlet function. The frequency samples at $\omega_k = \frac{2\pi k}{N}$ have the values

$$W_C[k] = \begin{cases} (1-a)N & k = 0 \\ \frac{aN}{2} & k = 1, N-1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find $W_C\left(\frac{5\pi}{N}\right)$ in terms of a and N, and then find the value of a that zeros it out, $W_C\left(\frac{5\pi}{N}\right) = 0$.

Note: in order to find the value of $W_C\left(\frac{5\pi}{N}\right)$, you will want to take advantage of the fact that, for small enough values of k,

$$\frac{\sin(k\pi/2)}{\sin(k\pi/2N)} \approx \frac{\sin(k\pi/2)}{k\pi/2N} = \begin{cases} \pm\frac{2N}{k\pi} & k \text{ odd} \\ 0 & k \text{ even and nonzero} \end{cases}$$

Solution:

$$W_C\left(\frac{5\pi}{N}\right) = \left(\frac{(1-a)2N}{5\pi} - \frac{a2N}{6\pi} - \frac{a2N}{14\pi}\right)$$

In order to zero it out, we need to find the value of a such that

$$0 = \frac{2(1-a)}{5} - \frac{a}{3} - \frac{a}{7}$$

Which gives us

$$1 = a\left(1 + \frac{5}{6} + \frac{5}{14}\right)$$

or $a = 0.4565217$. If we approximate this to 2 significant figures (i.e., if we tolerate an error of up to 0.01, which is -40dB), the approximation would be $a \approx 0.46$.