ECE 401 Signal and Image Analysis
Homework 3

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Assigned: 9/25/2023; Due: 10/4/2023
Reading: *DSP First* Chapter 5

Problem 3.1

In MP3, one of the filters you'll create is a local averaging filter. A local averaging filter produces an output $y[n]$, at time n, which is the average of the previous N samples of $x[m]$:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \tag{3.1-1}$$

$$h[m] = \begin{cases} \frac{1}{N} & 0 \le m \le N-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1-2}$$

(a) First, consider what happens if $x[m]$ is a pure tone with a period of $N_0 = \frac{2\pi}{\omega_0}$, an amplitude of A, and a phase of θ:

$$x[n] = A\cos(\omega_0 n - \theta)$$

Suppose that the averaging window, N, is exactly an integer multiple of N_0. For example, suppose that $N = 3N_0$. Draw a picture of $x[n]$ as a function of n, and shade in the regions that would be added together by the summation in Eq. (3.1-1) in order to compute $y[0]$. Argue based on your figure (with no calculations at all) that $y[0] = 0$.

Solution: The picture should show that we are adding together three complete periods of the cosine up through, and including, the sample $x[0]$. Every period of the cosine has a positive section and a negative section. When we average these two sections, they cancel each other out.

(b) Adding up the samples of a cosine is easy when N is an integer multiple of N_0, but hard otherwise. It's actually much easier to add the samples of a complex exponential, because we can use the standard geometric series formula (`https://en.wikipedia.org/wiki/Geometric_series#Formula`). Use that formula to find $y[N-1]$ when

$$x[n] = e^{j\omega_0 n}$$

Your result should have the form $y[0] = (1-a)/(1-b)$ for some complex-valued constants, a and b, that depend on π, N, and ω_0, but not on m or n.

Solution:

$$y[0] = \frac{1}{N}\left(\frac{1-e^{-j\omega_0 N}}{1-e^{-j\omega_0}}\right)$$

Problem 3.2

Another of the filters you'll create in MP3 is a backward-difference filter. A backward-difference filter is one of several different ways of approximating a first-derivative:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \tag{3.2-1}$$

$$h[m] = \begin{cases} 1 & m = 0 \\ -1 & m = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.2-2}$$

(a) First, consider what happens if $x[m]$ is a pure tone with a period of $T_0 = \frac{2\pi}{\omega_0}$, an amplitude of A, and a phase of θ:

$$x[n] = A\cos(\omega_0 n - \theta) \tag{3.2-3}$$

Plug Eq. (3.2-3) into Eq. (3.2-1), then use the following trigonometric identity and the following approximation to discover in exactly what way the first-difference operator approximates a derivative:

$$-2\sin(a)\sin(b) = \cos(a+b) - \cos(a-b)$$
$$\sin(b) \approx b \quad \text{if } b \text{ is small}$$

In order to apply the approximation, you can assume that ω_0 is a small number.

Solution: Plugging Eq. (3.2-3) into Eq. (3.2-1) gives

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= A\cos(\omega_0 n - \theta) - A\cos(\omega_0(n-1) - \theta) \end{aligned}$$

We can apply the trig identity if we set

$$\begin{aligned} a &= \omega_0(n - 0.5) - \theta \\ b &= 0.5\omega_0 \end{aligned}$$

which gives

$$y[n] = -2A\sin(0.5\omega_0)\sin(\omega_0(n-0.5) + \theta)$$

Applying the approximation gives

$$y[n] \approx -A\omega_0 \sin(\omega_0(n-0.5) + \theta),$$

which is what you would get if you differentiated $A\cos(\omega_0(n-0.5)+\theta)$ with respect to n.

(b) Let's try the same thing with a complex exponential. Plug the following value of $x[n]$ into Eq. (3.2-1)

$$x[n] = e^{j\omega_0 n},$$

then assume that ω_0 is a small number, and simplify using the approximation

$$e^{\phi} \approx 1 + \phi \quad \text{if } \phi \text{ is small}$$

in order to get something that looks like $dx[n]/dn$.

Solution:

$$\begin{aligned} y[n] &= x[n] - n[n-1] \\ &= e^{j\omega_0 n} - e^{j\omega_0(n-1)} \\ &= e^{j\omega_0 n}(1 - e^{-j\omega_0}) \\ &\approx j\omega_0 e^{-j\omega_0 n} \end{aligned}$$

which is $dx[n]/dn$.