UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
Fall 2023

EXAM 3

Wednesday, December 13, 2023, 7:00-10:00pm

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 7:00pm; you will need to photograph and upload your answers by exactly 10:00pm.
- There will be a total of 200 points in the exam (9 problems). Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

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Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Fourier Series

$$\textbf{Analysis:}\;\; X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$
$$\textbf{Synthesis:}\;\; x(t)=\sum_{k=-\infty}^{\infty}X_ke^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$
$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$
$$z_a=\begin{cases}z & f \bmod F_s<-f \bmod F_s\\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$
$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

Convolution

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

Frequency Response and DTFT

$$H(\omega)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$
$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(\omega)e^{j\omega n}d\omega$$
$$h[n]*\cos(\omega n)=|H(\omega)|\cos\left(\omega n+\angle H(\omega)\right)$$

Rectangular & Hamming Windows; Ideal LPF

$$w_R[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{otherwise}\end{cases}\leftrightarrow W_R(\omega)=e^{-\frac{j\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$
$$w_H[n]=0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right)w_R[n]\leftrightarrow W_H(\omega)=0.54W_R(\omega)-0.23W_R\left(\omega-\frac{2\pi}{N-1}\right)-0.23W_R\left(\omega+\frac{2\pi}{N-1}\right)$$
$$h_{\text{ideal}}[n]=\frac{\omega_c}{\pi}\text{sinc}(\omega_c n)\leftrightarrow H_{\text{ideal}}(\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & \text{otherwise}\end{cases}$$

Discrete Fourier Transform

$$\textbf{Analysis:}\;\; X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}$$
$$\textbf{Synthesis:}\;\; x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N}$$

Z Transform Pairs

$$b_k z^{-k} \leftrightarrow b_k \delta[n-k]$$
$$\frac{1}{1-az^{-1}} \leftrightarrow a^n u[n]$$
$$\frac{1}{(1-e^{-\sigma_1 - j\omega_1} z^{-1})(1-e^{-\sigma_1 + j\omega_1} z^{-1})} \leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1 (n+1)\right) u[n]$$

Notch Filter

$$H(z) = \frac{(1-e^{j\omega_1} z^{-1})(1-e^{-j\omega_1} z^{-1})}{(1-ae^{j\omega_1} z^{-1})(1-ae^{-j\omega_1} z^{-1})}$$
$$\mathrm{BW} = -2\ln(a)$$

1. (23 points) Suppose $x[n]$ is a real-valued signal that is nonzero only for $0 \leq n \leq N-1$, where N is an even number, and suppose $X(\omega)$ is its known DTFT. Suppose $y[n]$ is the periodic repetition of $x[n]$, i.e.,

$$y[n] = x\left[\langle n \rangle_N\right],$$

where $\langle n \rangle_N$ means "n modulo N." Notice that $y[n]$ can be written as

$$y[n] = \sum_{k=0}^{N/2} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)$$

Find A_k and θ_k in terms of $X(\omega)$, k, and N. Notice that your answers for $k=0$ and $k=\frac{N}{2}$ will be different than your answers for other values of k.

Solution:

$$A_k = \begin{cases} \frac{1}{N}|X(\frac{2\pi k}{N})| & k = 0, \frac{N}{2} \\ \frac{2}{N}|X(\frac{2\pi k}{N})| & \text{otherwise} \end{cases}$$

$$\theta_k = \angle X(\frac{2\pi k}{N})$$

2. (23 points) The system $\mathcal{R}$ computes the ratio of two consecutive samples:

$$\mathcal{R} : x[n] \to y[n] = \frac{x[n]}{x[n-1]}$$

(a) Is $\mathcal{R}$ linear? Prove your answer.

Solution:

$$x_1[n] \to y_1[n] = \frac{x_1[n]}{x_1[n-1]}$$
$$x_2[n] \to y_2[n] = \frac{x_2[n]}{x_2[n-1]}$$
$$x_3[n] = x_1[n] + x_2[n] \to y_3[n] = \frac{x_1[n] + x_2[n]}{x_1[n-1] + x_2[n-1]}$$
$$y_1[n] + y_2[n] = \frac{x_1[n]}{x_1[n-1]} + \frac{x_2[n]}{x_2[n-1]} \neq y_3[n]$$

So it is not linear.

(b) If $\mathcal{R}$ time-invariant? Prove your answer.

Solution:

$$x_1[n] \to y_1[n] = \frac{x_1[n]}{x_1[n-1]}$$
$$x_2[n] = x_1[n-m] \to y_2[n] = \frac{x_1[n-m]}{x_1[n-m-1]} = y_1[n-m]$$

So it is time-invariant.

3. (22 points) Suppose $h[n]$ is a length-L FIR lowpass filter with the following form:

$$h[n] = \begin{cases} \left(\frac{\omega_c}{\pi}\right) \operatorname{sinc}\left(\omega_c\left(n - \left(\frac{L-1}{2}\right)\right)\right) & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose the sampling rate is $F_s = 44000$ samples/second, and you want the transition band to be no more than 100Hz wide. What should L be?

Solution: L must be at least 440 samples.

4. (22 points) Consider the following signal:

$$x[n] = \begin{cases} 1 & 0 \le n \le 45 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the length-64 DFT of $x[n]$. Find $X[k]$ as a function of k.

Solution:

$$X[k] = e^{-j\frac{2\pi k 45}{2\times 64}} \frac{\sin\left(\frac{2\pi k 46}{2\times 64}\right)}{\sin\left(\frac{\pi k}{64}\right)}$$

5. (22 points) Consider the following signal:

$$x[n] = \begin{cases} 1 & 0 \le n \le 45 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the length-64 DFT of $x[n]$, $Y[k] = X^2[k]$, and $y[n]$ is the 64-point inverse DFT of $Y[k]$. Find $y[n]$ as a function of n.

Solution:

$$y[n] = \begin{cases} 28 & 0 \le n \le 26 \\ n+1 & 26 \le n \le 45 \\ 91-n & 45 \le n \le 63 \end{cases}$$

6. (22 points) Suppose you have a 30-minute recording, $x[n]$, that you want to filter using a lowpass filter $h[n]$ with a length of $L = 2011$. The most efficient way to find $H[k]$, the 4096-point DFT of $h[n]$, then compute

$$x_t[n] = \begin{cases} x[n + tM] & 0 \le n \le M - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X_t[k] = \sum_{n=0}^{M-1} x_t[n] e^{-j\frac{2\pi kn}{4096}}$$

$$Y_t[k] = X_k[k]H[k]$$

$$y_t[n] = \frac{1}{4096} \sum_{k=0}^{4095} Y_t[k] e^{j\frac{2\pi kn}{4096}}$$

$$y[n] = \sum_t y_t[n - tM]$$

Using the algorithm shown above, what is the largest value of M that will result in a value of $y[n]$ equal to $h[n] * x[n]$?

Solution: M must be less than or equal to $N - L + 1 = 2086$.

7. (22 points) Suppose $H(z)$ is given by

$$H(z) = \frac{1 - 0.9z^{-1}}{(1 - 0.4z^{-1})(1 + 0.4z^{-1})}$$

What is $h[n]$?

Solution:

$$h[n] = -\left(\frac{5}{8}\right)(0.4)^n u[n] + \left(\frac{13}{8}\right)(-0.4)^n u[n]$$

8. (22 points) Suppose you have a signal sampled at $F_s = 44,000$ samples/second, and you want to filter it to eliminate a narrowband noise signal at 1000Hz, with a 3dB bandwidth of 100Hz. Specify a filter $H(z)$ that will accomplish this task.

Solution:

$$H(z) = \frac{(1 - e^{j\frac{\pi}{22}} z^{-1})(1 - e^{-j\frac{\pi}{22}} z^{-1})}{(1 - e^{-\frac{\pi}{440}+j\frac{\pi}{22}} z^{-1})(1 - e^{-\frac{\pi}{440}-j\frac{\pi}{22}} z^{-1})}$$

9. (22 points) Consider the filter

$$y[n] = x[n] - a_1 y[n-1] - a_2 y[n-2]$$

The impulse response of this filter is $h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1))u[n]$. Use the quadratic formula $(\frac{-b \pm \sqrt{b^2-4ac}}{2a})$ to write explicit formulas for σ_1 and ω_1 in terms of a_1 and a_2. You may assume that $a_2 > a_1^2/4$.

Solution: The roots are

$$\begin{aligned} z_1, z_2 &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \\ &= e^{-\sigma_1 \pm j\omega_1} \end{aligned}$$

So

$$\begin{aligned} \sigma_1 &= \Re\{\ln\left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)\} \\ \omega_1 &= \Im\{\ln\left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)\} \end{aligned}$$

Alternate solution:

$$\begin{aligned} z_1 z_2 &= e^{-2\sigma_1} \\ &= \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)\left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}\right) \\ &= \frac{1}{4}(a_1^2 - (a_1^2 - 4a_2)) = a_2 \\ &\Rightarrow \sigma_1 = -\frac{1}{2}\ln a_2 \end{aligned}$$

$$\begin{aligned} z_1 + z_2 &= 2e^{-\sigma_1}\cos(\omega_1) \\ &= -a_1 \\ \cos(\omega_1) &= -\frac{a_1}{2}e^{\sigma_1} \\ &= -\frac{a_1}{2\sqrt{a_2}} \\ \omega_1 &= \cos^{-1}\left(-\frac{a_1}{2\sqrt{a_2}}\right) \end{aligned}$$

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