UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
Fall 2022

EXAM 3

Wednesday, December 14, 2022, 7:00-10:00pm

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 7:00pm; you will need to photograph and upload your answers by exactly 10:00pm.
- There will be a total of 200 points in the exam (9 problems). Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

netid: ______________________________

Phasors

$$A\cos(2\pi f t+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi f t}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi f t}+\frac{1}{2}e^{j\theta}e^{j2\pi f t}$$

Fourier Series

$$\textbf{Analysis:}\;\; X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\;\; x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s, -f \bmod F_s\right)$$

$$z_a=\begin{cases} z & f \bmod F_s < -f \bmod F_s \\ z^* & f \bmod F_s > -f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

Convolution

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

Frequency Response and DTFT

$$H(\omega)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(\omega)e^{j\omega n}d\omega$$

$$h[n]*\cos(\omega n)=|H(\omega)|\cos\left(\omega n+\angle H(\omega)\right)$$

Rectangular & Hamming Windows; Ideal LPF

$$w_R[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{otherwise}\end{cases}\leftrightarrow W_R(\omega)=e^{-\frac{j\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$w_H[n]=0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right)w_R[n]\leftrightarrow W_H(\omega)=0.54W_R(\omega)-0.23W_R\left(\omega-\frac{2\pi}{N-1}\right)-0.23W_R\left(\omega+\frac{2\pi}{N-1}\right)$$

$$h_{\text{ideal}}[n]=\frac{\omega_c}{\pi}\text{sinc}(\omega_c n)\leftrightarrow H_{\text{ideal}}(\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & \text{otherwise}\end{cases}$$

Discrete Fourier Transform

$$\textbf{Analysis:}\;\; X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}$$

$$\textbf{Synthesis:}\;\; x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N}$$

Z Transform Pairs

$$b_k z^{-k} \leftrightarrow b_k \delta[n-k]$$
$$\frac{1}{1-az^{-1}} \leftrightarrow a^n u[n]$$
$$\frac{1}{(1-e^{-\sigma_1 - \jmath\omega_1} z^{-1})(1-e^{-\sigma_1 + \jmath\omega_1} z^{-1})} \leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1 (n+1)\right) u[n]$$

1. (25 points) Suppose we have a continuous-time signal $x(t)$, given by

$$x(t) = (3+2j)e^{-j12\pi t/T_0} + (5-j)e^{-j6\pi t/T_0} + (5+j)e^{j6\pi t/T_0} + (3-2j)e^{j12\pi t/T_0}$$

(a) Let's multiply $x(t)$ by the cosine of $4\pi t/T_0$, and integrate over one period:

$$A = \int_0^{T_0} x(t)\cos\left(\frac{4\pi t}{T_0}\right) dt$$

What is A?

Solution:

$$\begin{aligned}
A &= \int_0^{T_0} x(t)\left(\frac{e^{j4\pi t/T_0} + e^{-j4\pi t/T_0}}{2}\right) dt \\
&= \int_0^{T_0} \left(\sum_k X_k e^{j\frac{2\pi k t}{T_0}}\right)\left(\frac{e^{j4\pi t/T_0} + e^{-j4\pi t/T_0}}{2}\right) dt \\
&= \frac{T_0}{2}\left(X_2 + X_{-2}\right) \\
&= 0
\end{aligned}$$

(b) Suppose we multiply, instead, by the cosine of $6\pi t/T_0$, and integrate over one period:

$$B = \int_0^{T_0} x(t)\cos\left(\frac{6\pi t}{T_0}\right) dt$$

What is B?

Solution:

$$\begin{aligned}
B &= \int_0^{T_0} x(t)\left(\frac{e^{j6\pi t/T_0} + e^{-j6\pi t/T_0}}{2}\right) dt \\
&= \int_0^{T_0} \left(\sum_k X_k e^{j\frac{2\pi k t}{T_0}}\right)\left(\frac{e^{j6\pi t/T_0} + e^{-j6\pi t/T_0}}{2}\right) dt \\
&= \frac{T_0}{2}\left(X_3 + X_{-3}\right) \\
&= \frac{T_0}{2}\left((5-j) + (5+j)\right) \\
&= 5T_0
\end{aligned}$$

(c) Continuing with the same $x(t)$: suppose we sample $x(t)$ with a sampling period of $T = T_0/8$, thus

$$y[n] = x(t)|_{t=nT}$$

The resulting $y[n]$ is periodic with a period of 8 samples, and with a discrete-time Fourier series of

$$y[n] = \sum_{k=-3}^{3} Y_k e^{j\frac{2\pi kn}{8}}$$

Find the values of Y_k, for $-3 \leq k \leq 3$.

Solution:

$$\begin{aligned} x[n] &= (3+2j)e^{-j12\pi nT/T_0} + (5-j)e^{-j6\pi nT/T_0} + (5+j)e^{j6\pi nT/T_0} + (3-2j)e^{j12\pi nT/T_0} \\ &= (3+2j)e^{-j12\pi n/8} + (5-j)e^{-j6\pi n/8} + (5+j)e^{j6\pi n/8} + (3-2j)e^{j12\pi n/8} \\ &= (3+2j)e^{j4\pi n/8} + (5-j)e^{-j6\pi n/8} + (5+j)e^{j6\pi n/8} + (3-2j)e^{-j4\pi n/8} \end{aligned}$$

Therefore,

$$\begin{aligned} Y_0 = Y_1 = Y_{-1} &= 0 \\ Y_2 &= 3+2j \\ Y_{-2} &= 3-2j \\ Y_3 &= 5+j \\ Y_{-3} &= 5-j \end{aligned}$$

2. (20 points) Consider a linear shift-invariant system with the following impulse response:

$$h[n] = \begin{cases} (0.9)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(a) Is this system stable? Why or why not?

Solution: Yes, because $\sum_{n=-\infty}^{\infty} |x[n]| = \frac{1}{1-0.9}$, which is finite.

(b) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Use convolution to find $y[n]$. You may find it useful to know that $\sum_{n=0}^{L-1} a^n = \frac{1-a^L}{1-a}$.

Solution: When $n \leq 9$,

$$\begin{aligned} y[n] &= \sum_{m=0}^{n} (0.9)^{n-m} \\ &= (0.9)^n \sum_{m=0}^{n} (0.9)^{-m} \\ &= (0.9)^n \sum_{m=0}^{n} \left(\frac{10}{9}\right)^m \\ &= (0.9)^n \frac{1 - \left(\frac{10}{9}\right)^{n+1}}{1 - \left(\frac{10}{9}\right)} \\ &= \frac{(0.9)^n - \left(\frac{1}{0.9}\right)}{1 - \left(\frac{1}{0.9}\right)} \end{aligned}$$

When $n \geq 10$,

$$\begin{aligned} y[n] &= \sum_{m=0}^{9} (0.9)^{n-m} \\ &= (0.9)^n \sum_{m=0}^{9} (0.9)^{-m} \\ &= (0.9)^n \frac{1 - \left(\frac{10}{9}\right)^{10}}{1 - \left(\frac{10}{9}\right)} \end{aligned}$$

Thus

$$y[n] = \begin{cases} 0 & n < 0 \\ (0.9)^n \frac{1-\left(\frac{10}{9}\right)^{n+1}}{1-\left(\frac{10}{9}\right)} & 0 \leq n \leq 9 \\ (0.9)^n \frac{1-\left(\frac{10}{9}\right)^{10}}{1-\left(\frac{10}{9}\right)} & 10 \leq n \end{cases}$$

3. (25 points) A linear shift-invariant system has the following frequency response:

$$H(\omega) = e^{-5j\omega}\left(1 + \frac{1}{2}\cos\omega\right)$$

(a) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = 3\sin\left(\frac{\pi n}{6}\right)$$

What is $y[n]$?

Solution:

$$y[n] = 3\left(1 + \frac{1}{2}\cos(\pi/6)\right)\sin\left(\frac{\pi(n-5)}{6}\right)$$

(b) What is the impulse response, $h[n]$, of this system?

Solution:

$$h[n] = \frac{1}{4}\delta[n-4] + \delta[n-5] + \frac{1}{4}\delta[n-6]$$

4. (20 points) Suppose you want an FIR bandpass filter with a length of $N = 1024$, with cutoff frequencies $\omega_{\text{LO}} = 0.46\pi$ and $\omega_{\text{HI}} = 0.48\pi$, and with little stopband ripple. Find an impulse response $h[n]$ that meets these requirements.

Solution: The solution is the difference of two ideal lowpass filters, shifted by an odd number of half-samples (e.g., $\frac{1023}{2}$), then windowed so that it is symmetric and has 1024 samples (e.g., $0 \le n \le 1023$). To minimize stop-band ripple, it should be windowed by some tapered window, thus:

$$h[n] = w[n]\left(0.48\text{sinc}\left(0.48\pi\left(n - \frac{1023}{2}\right)\right) - 0.46\text{sinc}\left(0.46\pi\left(n - \frac{1023}{2}\right)\right)\right),$$

where $w[n]$ is a tapered window such as a Hamming, Hann, or Bartlett window, e.g.,

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{1023}\right) & 0 \le n \le 1023 \\ 0 & \text{otherwise} \end{cases}$$

5. (25 points) Consider the signal

$$x[n] = \cos\left(\frac{\pi n}{100}\right)$$

(a) What is the 128-sample DFT, $X[k]$, of this signal? Be sure to consider windowing effects.

Solution:

$$w[n]x[n] = \frac{1}{2}w[n]e^{j\pi n/100} + \frac{1}{2}w[n]e^{-j\pi n/100}$$

$$\text{DTFT}\left\{w[n]x[n]\right\} = \frac{1}{2}W\left(\omega - \frac{\pi}{100}\right) + \frac{1}{2}W\left(\omega + \frac{\pi}{100}\right),$$

where $w[n]$ is a rectangular window, so

$$W(\omega) = e^{-j\omega\frac{127}{2}}\frac{\sin(64\omega)}{\sin(\omega/2)}$$

$$\begin{aligned} X[k] &= \text{DTFT}\left\{w[n]x[n]\right\}\big|_{\omega=\frac{2\pi k}{128}} \\ &= \frac{1}{2}W\left(\frac{2\pi k}{128} - \frac{\pi}{100}\right) + \frac{1}{2}W\left(\frac{2\pi k}{128} + \frac{\pi}{100}\right) \end{aligned}$$

(b) Suppose you want to construct the following periodic signal:

$$y[n] = \cos\left(\frac{\pi(n - 128\ell)}{100}\right), \quad 128\ell \leq n < 128(\ell+1), \quad \text{for all } \ell$$

You can create this signal using the following Fourier series:

$$y[n] = \sum_{k=0}^{127} Y_k e^{j\frac{2\pi kn}{128}}$$

Notice that there is a relationships between Y_k and the DFT coefficients, $X[k]$, that you computed in part (a) of this problem. Find Y_k in terms of $X[k]$.

Solution:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
Y_k &= \frac{1}{N}\sum_{n=0}^{N-1} y[n]e^{-j\frac{2\pi kn}{N}} \\
&= \frac{1}{N}X[k] \\
&= \frac{1}{128}X[k]
\end{aligned}$$

6. (25 points) Suppose you have a very long signal, $x[n]$, that you want to filter to compute $y[n] = h[n]*x[n]$. This can be done using the following sequence of steps:

$$H[k] = \sum_{n=0}^{N-1} h[n] e^{-j\frac{2\pi kn}{N}} \tag{1}$$

$$x_\ell[n] = \begin{cases} x[n+\ell M] & 0 \le n \le M-1, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$X_\ell[k] = \sum_{n=0}^{N-1} x_\ell[n] e^{-j\frac{2\pi kn}{N}} \tag{3}$$

$$Y_\ell[k] = H[k] X_\ell[k] \tag{4}$$

$$y_\ell[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y_\ell[k] e^{j\frac{2\pi kn}{N}} \tag{5}$$

$$y[n] = \sum_{\ell=-\infty}^{\infty} y_\ell[n-\ell M] \tag{6}$$

(a) Suppose $h[n]$ is 129 samples long. Find values of M and N such that the algorithm in Eqs. (1) through (6) gives the same result as $y[n] = h[n] * x[n]$. There are many different correct answers; you only need to find one correct answer.

Solution: Any pair of values such that $N \ge L + M - 1$ is a valid answer. For example, you could say that $M = 128$ and $N = 256$.

(b) Suppose $x[n] = \cos(0.08\pi n)$, and $h[n] = \delta[n-126]$. Find $x_\ell[n]$ and $y_\ell[n]$ in terms of ℓ, M, and n.

Solution:

$$x_\ell[n] = \begin{cases} \cos(0.08\pi(n+\ell M)) & 0 \le n \le M-1, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} y_\ell[n] &= h[n] \circledast x_\ell[n] \\ &= x_\ell\left[\langle n-126\rangle_N\right] \\ &= \begin{cases} \cos(0.08\pi(n+\ell M - 126)) & 126 \le n \le M+125, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

7. (20 points) Suppose that

$$H(z) = \frac{1}{1 - 0.9e^{j\pi/6}z^{-1}} + \frac{1}{1 - 0.9e^{-j\pi/6}z^{-1}}$$

Find the pole(s) and zero(s) of $H(z)$.

Solution:

$$\begin{aligned} H(z) &= \frac{(1 - 0.9e^{j\pi/6}z^{-1}) + (1 - 0.9e^{-j\pi/6}z^{-1})}{(1 - 0.9e^{j\pi/6}z^{-1})(1 - 0.9e^{-j\pi/6}z^{-1})} \\ &= \frac{2 - 1.8\cos(\pi/6)z^{-1}}{(1 - 0.9e^{j\pi/6}z^{-1})(1 - 0.9e^{-j\pi/6}z^{-1})} \qquad = 2\frac{1 - 0.9\cos(\pi/6)z^{-1}}{(1 - 0.9e^{j\pi/6}z^{-1})(1 - 0.9e^{-j\pi/6}z^{-1})} \end{aligned}$$

This has one zero, r_1, and two poles, p_1 and p_2:

$$\begin{aligned} r_1 &= 0.9\cos(\pi/6) = 0.45\sqrt{3} \\ p_1 &= 0.9e^{j\pi/6} \\ p_2 &= 0.9e^{-j\pi/6} \end{aligned}$$

8. (20 points) Consider a notch filter with zeros at $r_1 = e^{j0.47\pi}$ and $r_2 = e^{-j0.47\pi}$, and with poles at $p_1 = 0.999e^{j0.47\pi}$ and $p_2 = 0.999e^{-j0.47\pi}$.

(a) What is the 3dB bandwidth of the notch, expressed in radians per sample?

> **Solution:** The bandwidth is $2\sigma = -2\ln(0.999)$.

(b) This filter can be implemented as

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Find b_1, b_2, a_1, and a_2.

> **Solution:**
>
> $$\begin{aligned} H(z) &= \frac{(1 - e^{j0.47\pi}z^{-1})(1 - e^{-j0.47\pi}z^{-1})}{(1 - 0.999e^{j0.47\pi}z^{-1})(1 - 0.999e^{-j0.47\pi}z^{-1})} \\ &= \frac{1 - 2\cos(0.47\pi)z^{-1} + z^{-2}}{1 - 1.998\cos(0.47\pi)z^{-1} + (0.999)^2 z^{-2}} \\ &= \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$
>
> So,
>
> $$\begin{aligned} b_1 &= -2\cos(0.47\pi) \\ b_2 &= 1 \\ a_1 &= -1.998\cos(0.47\pi) \\ a_2 &= (0.999)^2 \end{aligned}$$

9. (20 points) A particular bell has a resonance at 440Hz, with a decay time of half a second, and another resonance at 1320Hz, with a decay time of 3 seconds. Find a filter, $H(z)$, whose impulse response sounds like the impulse response of this bell if played through a D/A at $F_s = 10,000$ samples per second.

Solution: The decay time specifies the filter bandwidths; we have that $e^{-\sigma n} = e^{-1}$ when $n =$decay time$\times$sampling rate. In this case,

$$\begin{aligned}
\sigma_1 &= \frac{1}{0.5 \times 10000} = \frac{1}{5000} \\
\sigma_2 &= \frac{1}{3 \times 10000} = \frac{1}{30000} \\
\omega_1 &= \frac{2\pi 440}{10000} \\
\omega_2 &= \frac{2\pi 1320}{10000}
\end{aligned}$$

The filter can actually be either a parallel or a series connection of these two resonators. If it's a parallel connection, it would be

$$\begin{aligned}
H(z) &= \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} + \frac{1}{(1 - p_2 z^{-1})(1 - p_2^* z^{-1})} \\
p_1 &= e^{-n/5000} e^{j\frac{2\pi 440}{10000}} \\
p_2 &= e^{-n/30000} e^{j\frac{2\pi 1320}{10000}}
\end{aligned}$$