UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2021

# PRACTICE EXAM 3

Exam 3 will be held Tuesday, December 14, 8:00-11:00am

- This will a CLOSED BOOK exam.
- You will be permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers will not be permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will be sent to you by e-mail and on zoom at exactly 8:00am; you will need to photograph and upload your answers by exactly 11:00am.
- There will be a total of 200 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ________________________________________

1. (20 points) Consider the following signal:

$$x[n] = \delta[n-15] + \delta[n-30]$$

(a) $X[k]$ is the 32-point DFT of $x[n]$. Specify $X[k]$ as a function of $k$.

**Solution:**

$$X[k] = e^{-j\frac{30\pi k}{32}} + e^{-j\frac{60\pi k}{32}}$$

(b) Suppose that $h[n]$ is defined as follows:

$$h[n] = \begin{cases} e^{-n/14} & 0 \le n \le 14 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $H[k]$ is the 32-point DFT of $h[n]$, $Y[k] = H[k]X[k]$, and $y[n]$ is the inverse DFT of $Y[k]$. Find $y[n]$.

**Solution:**

$$y[n] = \begin{cases} e^{-(n+2)/14} & 0 \le n \le 12 \\ e^{-(n-15)/14} & 15 \le n \le 29 \\ e^{-(n-30)/14} & 30 \le n \le 31 \end{cases}$$

2. (20 points) Consider the following system. The input of this system is $x[n]$, and the output is $y[n]$:

$$v[n] = x[n] + 0.9v[n-1]$$
$$y[n] = v[n] - 0.7y[n-1]$$

(a) What is the system function, $H(z)$, for this system?

**Solution:**

$$H(z) = \frac{1}{(1 - 0.9z^{-1})(1 + 0.7z^{-1})}$$

(b) What is the impulse response of this system?

**Solution:**

$$\begin{aligned}
\frac{1}{(1 - 0.9z^{-1})(1 + 0.7z^{-1})} &= \frac{C_1}{1 - 0.9z^{-1}} + \frac{C_2}{1 + 0.7z^{-1}} \\
1 &= C_1(1 + 0.7z^{-1}) + C_2(1 - 0.9z^{-1}) \\
1 &= C_1(1 + 0.7/0.9) = C_1(16/9) \\
1 &= C_2(1 - 0.9/(-0.7)) = C_2(16/7)
\end{aligned}$$

So $C_1 = \frac{9}{16}$, $C_2 = \frac{7}{16}$, and

$$h[n] = \left(\frac{9}{16}\right)(0.9)^n u[n] + \left(\frac{7}{16}\right)(-0.7)^n u[n]$$

3. (20 points) A flute player is playing a middle-A note. This system can be well modeled by blowing white noise through a damped resonator with a resonant frequency of 440Hz, and a bandwidth of 20Hz. Suppose you want to synthesize this flute digitally, by blowing white noise through a second-order damped resonator, with a sampling frequency of $F_s = 10,000$Hz.

(a) You want to implement the resonator as

$$y[n] = x[n] + a_1 y[n-1] + a_2 y[n-2]$$

What are $a_1$ and $a_2$?

**Solution:** The resonant frequency and half-bandwidth are

$$\omega_1 = \frac{2\pi 440}{10,000}$$
$$\sigma_1 = \frac{2\pi 10}{10,000}$$

So the system function is

$$H(z) = \frac{1}{\left(1 - e^{-\frac{2\pi 10}{10000} + j\frac{2\pi 440}{10000}} z^{-1}\right)\left(1 - e^{-\frac{2\pi 10}{10000} - j\frac{2\pi 440}{10000}} z^{-1}\right)}$$

$a_1 = (p_1 + p_1^*)$, and $a_2 = -|p_1|^2$, so

$$a_1 = 2e^{-\frac{2\pi 10}{10000}} \cos\left(\frac{2\pi 440}{10000}\right)$$
$$a_2 = -e^{-\frac{4\pi 10}{10000}}$$

(b) Suppose you have succeeded in implementing the digital filter. Now you give it the input $x[n] = \delta[n]$. What is the output?

**Solution:** The output is $h[n]$, which is

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1 (n+1)) u[n]$$

where

$$\omega_1 = \frac{2\pi 440}{10,000}$$
$$\sigma_1 = \frac{2\pi 10}{10,000}$$

4. (20 points) You have recorded an electrocardiogram signal, $x[n]$, with a sampling frequency of $F_s = 1.2kHz$. Unfortunately, it has been corrupted by power line noise: it has a big sinusoidal component at 60Hz. Fortunately, you know how to eliminate power line noise using a notch filter. All you have to do is to pass the signal through a difference equation:

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2] \tag{1}$$

Use a pole amplitude of 0.98. What are $b_1$, $b_2$, $a_1$, and $a_2$?

(Note: leave your answer in the form of an explicit numerical expression. For example, if you discover that $a_1 = (0.3)^2 \sin(2\pi/400)$, then you should leave it in that form instead of trying to simplify.)

**Solution:**

$$\begin{aligned} b_1 &= -2\cos(2\pi 60/1200) \\ b_2 &= 1 \\ a_1 &= -2(0.98)\cos(2\pi 60/1200) \\ a_2 &= (0.98)^2 \end{aligned}$$

5. (10 points) Consider the signal $x(t) = -2 + \sin(40\pi t)$. Suppose we sample this signal at $F_s = 100\text{Hz}$, then take a length-20 DFT of 20 samples of this signal. For which values of $k$, $0 \leq k \leq 19$, will the DFT samples $X[k]$ be nonzero?

**Solution:** The discrete-time signal will be

$$x[n] = -2 + \sin\left(\frac{40\pi n}{100}\right) = -2 + \sin(0.4\pi n)$$

The DTFT is nonzero at frequencies $\omega \in \{0, 0.4\pi, 1.6\pi\}$. The DTFT computes samples at $\omega_k = \frac{2\pi k}{20} = 0.1\pi k$, so it is nonzero for $k \in \{0, 4, 16\}$.

6. (20 points) Determine whether the following LTI system is causal and/or BIBO stable:

$$y[n] = x[n+1] + y[n-1]$$

(a) Is it causal? Describe your reasoning in words.

**Solution:** No. $y[n]$ depends on $x[n+1]$, which is in the future, therefore this system is not causal.

(b) Is it stable? Prove your answer.

**Solution:** No. The system function is

$$H(z) = \frac{z}{1 - z^{-1}}$$

There is a pole at $z = 1$, which does not meet the criterion $|z| < 1$, therefore it is unstable.

7. (20 points) A particular system generates an output $y[n]$ from its input $x[n]$ according to the following rule:

$$y[n] = \begin{cases} x[n] & n \text{ is even} \\ \frac{1}{2}\left(x[n-1] + x[n+1]\right) & n \text{ is odd} \end{cases}$$

(a) Is the system causal? Give your reason.

**Solution:** No. When $n$ is odd, $y[n]$ is the average of $x[n-1]$ (which is in the past) and $x[n+1]$ (which is in the future). Since it depends on a future input, it is not causal.

(b) Is the system stable? Give your reason.

**Solution:** Yes. For every finite input signal ($|x[n]| \leq M$ for some finite $M$), the corresponding output signal is also finite ($|y[n]| \leq M$ for the same $M$), so the system is stable.

8. (15 points) A signal is corrupted with narrowband noise at $\omega_n = \frac{\pi}{6}$ radians/sample. To remove the noise, you create a notch filter with a pole magnitude of $r = 0.9$. The filter system function is

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

Speciy the poles and zeros in magnitude/phase form (e.g., $p_1 = me^{j\theta}$, but you should specify the numerical value of $m$ and the numerical value of $\theta$).

**Solution:**

$$r_1 = e^{j\pi/6}$$

$$r_2 = e^{-j\pi/6}$$

$$p_1 = 0.9e^{j\pi/6}$$

$$p_2 = 0.9e^{-j\pi/6}$$

9. (15 points) Suppose you want to implement a filter with the following frequency response:

$$H(\omega) = \frac{(1 - 0.5e^{j\pi/4}e^{-j\omega})(1 - 0.5e^{-j\pi/4}e^{-j\omega})}{(1 + 0.5e^{j\pi/4}e^{-j\omega})(1 + 0.5e^{-j\pi/4}e^{-j\omega})}$$

You realize that you can get this frequency response by writing one line of code in matlab, implementing the following equation:

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$

Find the filter coefficients.

**Solution:**

$$a_1 = -\frac{\sqrt{2}}{2}$$

$$a_2 = -\frac{1}{4}$$

$$b_1 = -\frac{\sqrt{2}}{2}$$

$$b_2 = \frac{1}{4}$$

10. (30 points) Suppose you have a signal sampled at $F_s = 600$ samples/second. You wish to create a notch filter to eliminate a noise component at 60Hz. You choose to do this using the following filter:

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

(a) Specify the values of the poles $p_1, p_2$ and the zeros $r_1, r_2$. Note that there is one free parameter in your answer that is not specified by the problem statement; you may set that free parameter to any reasonable value.

**Solution:** $r_1 = e^{j\pi/5}$, $r_2 = e^{-j\pi/5}$, $p_1 = 0.99e^{j\pi/5}$, $p_2 = 0.99e^{-j\pi/5}$

(b) Now suppose you are given a system function

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

and you wish to implement this using the equation

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$

Find $b_1$, $b_2$, $a_1$ and $a_2$ in terms of $r_1$, $r_2$, $p_1$ and $p_2$.

**Solution:** $b_1 = -(r_1 + r_2)$, $b_2 = r_1 r_2$, $a_1 = (p_1 + p_2)$, $a_2 = -p_1 p_2$