UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2021

EXAM 3

Tuesday, December 14, 2021, 8:00-11:00am

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 200 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ____

netid: ____

Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases}z & f \bmod F_s<-f \bmod F_s\\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

Convolution

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

Frequency Response and DTFT

$$H(\omega)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(\omega)e^{j\omega n}d\omega$$

$$h[n]*\cos(\omega n)=|H(\omega)|\cos\left(\omega n+\angle H(\omega)\right)$$

Rectangular & Hamming Windows; Ideal LPF

$$w_R[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{otherwise}\end{cases}\leftrightarrow W_R(\omega)=e^{-\frac{j\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$w_H[n]=0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right)w_R[n]\leftrightarrow W_H(\omega)=0.54W_R(\omega)-0.23W_R\left(\omega-\frac{2\pi}{N-1}\right)-0.23W_R\left(\omega+\frac{2\pi}{N-1}\right)$$

$$h_{\text{ideal}}[n]=\frac{\omega_c}{\pi}\text{sinc}(\omega_c n)\leftrightarrow H_{\text{ideal}}(\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & \text{otherwise}\end{cases}$$

Discrete Fourier Transform

$$\textbf{Analysis:}\ \ X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}$$

$$\textbf{Synthesis:}\ \ x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N}$$

Z Transform Pairs

$$b_k z^{-k} \leftrightarrow b_k \delta[n-k]$$

$$\frac{1}{1-az^{-1}} \leftrightarrow a^n u[n]$$

$$\frac{1}{(1-e^{-\sigma_1 - \jmath\omega_1} z^{-1})(1-e^{-\sigma_1 + \jmath\omega_1} z^{-1})} \leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1 (n+1)\right) u[n]$$

1. (17 points) The variables A and θ are defined by the equation

$$4\cos\left(2600\pi t + \frac{\pi}{3}\right) + 3\sin\left(2600\pi t\right) = A\cos\left(2600\pi t + \theta\right)$$

Find explicit numerical expressions for A and θ.

Solution:

$$\begin{aligned} A &= \sqrt{(4\cos(\pi/3) + 3\cos(-\pi/2))^2 + (4\sin(\pi/3) + 3\sin(-\pi/2))^2} \\ &= \sqrt{25 - 12\sqrt{3}} \\ \theta &= \mathrm{atan}\left(\frac{4\sin(\pi/3) + 3\sin(-\pi/2)}{4\cos(\pi/3) + 3\cos(-\pi/2)}\right) \\ &= \mathrm{atan}\left(\frac{2\sqrt{3} - 3}{2}\right) \end{aligned}$$

2. (16 points) $x(t)$ and $y(t)$ are two signals that are both periodic with fundamental frequency F_0, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}, \qquad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j2\pi k F_0 t}$$

The relationship between $x(t)$ and $y(t)$ is given by

$$y(t) = x(t - 0.001) + 3\frac{dx}{dt}$$

Find Y_k in terms of X_k and F_0.

Solution:

$$Y_k = X_k \left(e^{-0.001j2\pi k F_0} + 3j2\pi k F_0\right)$$

3. (17 points) A zebra is photographed at a resolution such that each black stripe is 3.5 pixels wide, and each white strip is 3.5 pixels wide. Modeling this as a square wave, we get the following model:

$$x[n] = 128 + 81\cos\left(\frac{\pi n}{7}\right) + 27\cos\left(\frac{3\pi n}{7}\right)$$

Unfortunately, the lens has some blur, so the image that is actually recorded is

$$y[n] = \frac{3}{4}x[n] + \frac{1}{4}x[n-1] \tag{1}$$

For some appropriate values of A_1, θ_1, A_3, and θ_3, $y[n]$ is equal to

$$y[n] = 128 + A_1\cos\left(\frac{\pi n}{7} + \theta_1\right) + A_3\cos\left(\frac{3\pi n}{7} + \theta_3\right) \tag{2}$$

Find explicit numerical expressions for A_1, θ_1, A_3, and θ_3 such that Eq. (2) is equal to Eq. (1).

Solution:

$$H(\omega) = \frac{3}{4} + \frac{1}{4}e^{-j\omega}$$
$$A_1 = 81|H(\pi/7)| = \frac{81}{4}\sqrt{(3 + cos(\pi/7))^2 + \sin^2(\pi/7)}$$
$$A_3 = 27|H(3\pi/7)| = \frac{27}{4}\sqrt{(3 + cos(3\pi/7))^2 + \sin^2(3\pi/7)}$$
$$\theta_1 = \angle H(\pi/7) = -\text{atan}\left(\frac{\sin(\pi/7)}{3 + cos(\pi/7)}\right)$$
$$\theta_3 = \angle H(3\pi/7) = -\text{atan}\left(\frac{\sin(3\pi/7)}{3 + cos(3\pi/7)}\right)$$

4. (16 points) It's hard to convolve two sinc functions, but it's easy to muliply their Fourier transforms. For example, suppose that

$$x[n] = \text{sinc}(0.5\pi n)$$
$$h[n] = \frac{1}{3}\text{sinc}\left(\frac{\pi n}{3}\right)$$
$$y[n] = x[n] * h[n]$$

Find $y[n]$ as a function of n; x and h should not appear in your answer.

Solution:

$$y[n] = \frac{2}{3}\text{sinc}\left(\frac{\pi n}{3}\right)$$

5. (17 points) Consider the signal

$$x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

Find $X[k]$, the 4-point DFT of $x[n]$. The variable k should not appear in your answer; instead, please write explicit numerical expressions for the four samples of $X[k]$, $k \in \{0, 1, 2, 3\}$.

Solution:

$$\begin{aligned}
X[0] &= 6 \\
X[1] &= 3 + 2e^{-j\pi/2} + e{-}j\pi = 2 - 2j \\
X[2] &= 3 + 2e^{-j2\pi/2} + e^{-j2\pi} = 2 \\
X[3] &= 3 + 2e^{-j3\pi/2} + e^{-j3\pi} = 2 + 2j
\end{aligned}$$

6. (17 points) Suppose $x[n] = \cos(0.32\pi n) w_H[n]$, where $w_H[n]$ is a nearly-Hamming window defined as follows:

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{100}\right) & 0 \le n \le 99 \\ 0 & \text{otherwise} \end{cases}$$

Let $X[k]$ be the 100-point DFT of $x[n]$. For which values of k, $0 \le k \le 99$, is $X[k]$ nonzero?

Solution:

$$\begin{aligned} X[k] &= X(\omega) \text{ at } \omega = \frac{2\pi k}{100} \\ &= W_H[k-16] + W_H[k+16] \\ &= W_H[k-16] + W_H[k-100+16] \end{aligned}$$

The normal $W_H[k]$ is nonzero everywhere, because the raised cosine has a frequency of $\frac{2\pi}{N-1}$. This modified Hamming window, however, has a frequency of $\frac{2\pi}{N}$, so we get that

$$W_H[k] = 0.54 W_R[k] + 0.23 W_R[k-1] + W_R[k+1]$$

Since $W_R[k]$ is nonzero only at $k = 0$, we have that $X[k]$ is nonzero only at

$$k \in \{15, 16, 17, 100-17, 100-16, 100-15\}$$

7. (16 points) Suppose

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 15 \text{ or } 49 \leq n \leq 63 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X[k]$ is the 64-point DFT of $x[n]$, and

$$Y[k] = e^{-j\pi k} X[k]$$

Find $y[n]$, the inverse DTFT of $Y[k]$.

Solution:

$$y[n] = \begin{cases} 1 & 49 + 32 - 64 \leq n \leq 15 + 32 \\ 0 & \text{otherwise} \end{cases}$$

8. (17 points) Suppose $h[n]$ is a periodically repeated, rectangular-windowed ideal lowpass filter of length 94 samples

$$h[n] = \begin{cases} \frac{1}{3}\text{sinc}\left(\frac{\pi n}{3}\right) & 0 \le n \le 46 \\ \frac{1}{3}\text{sinc}\left(\frac{\pi}{3}(n-94)\right) & 47 \le n \le 93 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $x[n] = \delta[n-15]$, $X[k]$ is the DFT of $x[n]$, and $Y[k] = H[k]X[k]$. Find $y[n]$, the inverse DFT of $Y[k]$.

Solution:

$$y[n] = \begin{cases} \frac{1}{3}\text{sinc}\left(\frac{\pi(n-15)}{3}\right) & 0 \le n \le 61 \\ \frac{1}{3}\text{sinc}\left(\frac{\pi}{3}(n-109)\right) & 62 \le n \le 93 \\ 0 & \text{otherwise} \end{cases}$$

9. (17 points) Consider the difference equation

$$y[n] = x[n] - 0.6x[n-1] + 0.2x[n-2]$$

Find explicit numerical expressions for the frequencies, $\omega_1 = \angle z_1$ and $\omega_2 = \angle z_2$, of the zeros of this filter.

Solution:

$$\omega_1 = \text{atan}\left(\frac{\sqrt{0.8 - 0.36}}{0.6}\right)$$
$$\omega_2 = -\text{atan}\left(\frac{\sqrt{0.8 - 0.36}}{0.6}\right)$$

10. (16 points) Suppose you have designed a filter with the following system function:

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.7z^{-1}}$$

What is $h[n]$, the impulse response of this filter?

Solution:

$$h[n] = (0.7)^n u[n] + 0.5(0.7)^{n-1} u[n-1]$$

11. (17 points) A recording of a conversation, at $F_s = 16,000$ samples/second, is corrupted by an additive pure tone at exactly $f_1 = 1000$Hz. The tone is quite narrowband, so you can get rid of it using a notch filter with a 20Hz bandwidth. Write the difference equation for this notch filter; specify all coefficients as numbers, or as explicit numerical expressions.

Solution:

$$y[n] = x[n] - 2\cos\left(\frac{\pi}{8}\right)x[n-1] + x[n-2] + 2e^{-\frac{\pi}{800}}\cos\left(\frac{\pi}{8}\right)y[n-1] - e^{-\frac{2\pi}{800}}y[n-2]$$

12. (17 points) A particular signal has the following z-transform:

$$X(z) = \frac{1}{1 - 0.9e^{0.1\pi j} z^{-1}}$$

The signal, $x[n]$, is complex-valued in the time domain, with an exponentially decreasing absolute value. Find an explicit numerical expression for the smallest integer, n, such that

$$\frac{|x[n]|}{|x[0]|} \le \frac{1}{e},$$

where e is the base of the natural logarithm.

Solution:

$$n \ge -\frac{1}{\ln(0.9)}$$