

# Lecture 34: Final Exam Review

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis

- 1 Topics
- 2 DFT
- 3 Circular Convolution
- 4 Z Transform
- 5 Autoregressive Filters
- 6 Inverse Z Transform
- 7 Notch Filters
- 8 Resonators
- 9 Summary

# Outline

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# Final Exam: General Structure

- About twice as long as a midterm (i.e., 8-10 problems with 1-3 parts each)
- You'll have 3 hours for the exam
- The usual rules: no calculators or computers, two sheets of handwritten notes, you will have two pages of formulas provided on the exam, published by the Friday before the exam.

# Final Exam: Topics Covered

- 17%: Material from exam 1 (phasors, Fourier series)
- 17%: Material from exam 2 (LSI systems, DTFT)
- 66%: Material from the last third of the course (DFT, Z transform)

# Material from the last third of the course

- DFT & Window Design
- Circular Convolution
- Z Transform & Inverse Z Transform
- Notch Filters & Second-Order IIR

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# DFT and Inverse DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$



# DFT of a Cosine

$$x[n] = \cos(\omega_0 n)w[n] \leftrightarrow X(\omega_k) = \frac{1}{2}W(\omega_k - \omega_0) + \frac{1}{2}W(\omega_k + \omega_0)$$

where  $W(\omega)$  is the transform of  $w[n]$ . For example, if  $w[n]$  is a rectangular window, then

$$W(\omega) = e^{-j\omega \frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

# Properties of the DFT

- The DFT is periodic in frequency:

$$X[k + N] = X[k]$$

- The inverse DFT is periodic in time: if  $x[n]$  is the inverse DFT of  $X[k]$ , then

$$x[n + N] = x[n]$$

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k]$$

- Samples of the DTFT: if  $x[n]$  is finite in time, with length  $\leq N$ , then

$$X[k] = X(\omega_k), \quad \omega_k = \frac{2\pi k}{N}$$

# Properties of the DFT

- Conjugate symmetric:

$$X[k] = X^*[-k] = X^*[N - k]$$

- Frequency shift:

$$w[n]e^{j\frac{2\pi k_0 n}{N}} \leftrightarrow W[k - k_0]$$

- Circular time shift:

$$x[\langle n - n_0 \rangle_N] \leftrightarrow e^{j\frac{2\pi kn_0}{N}} X[k]$$

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# DFT is actually a Fourier Series

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j \frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi kn}{N}}$$

# Circular Convolution

$$Y[k] = H[k]X[k]$$

$$y[n] = h[n] \circledast x[n]$$

$$= \sum_{m=0}^{N-1} h[m] x[\langle n - m \rangle_N]$$

$$= \sum_{m=0}^{N-1} x[m] h[\langle n - m \rangle_N]$$

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# Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$



# System Function

$$y[n] = 0.2x[n+3] + 0.3x[n+2] + 0.5x[n+1] \\ - 0.5x[n-1] - 0.3x[n-2] - 0.2x[n-3]$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.2z^3 + 0.3z^2 + 0.5z^1 - 0.5z^{-1} - 0.3z^{-2} - 0.2z^{-3}$$

# The Zeros of $H(z)$

- The roots,  $z_1$  and  $z_2$ , are the values of  $z$  for which  $H(z) = 0$ .
- But what does that mean? We know that for  $z = e^{j\omega}$ ,  $H(z)$  is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the roots do not have unit magnitude:

$$z_1 = 1 + j = \sqrt{2}e^{j\pi/4}$$

$$z_2 = 1 - j = \sqrt{2}e^{-j\pi/4}$$

- What it means is that, when  $\omega = \frac{\pi}{4}$  (so  $z = e^{j\pi/4}$ ), then  $|H(\omega)|$  is as close to a zero as it can possibly get. So at that frequency,  $|H(\omega)|$  is as low as it can get.

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## General form of an FIR filter

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

This filter has an impulse response ( $h[n]$ ) that is  $M + 1$  samples long.

- The  $b_k$ 's are called **feedforward** coefficients, because they feed  $x[n]$  forward into  $y[n]$ .

## General form of an IIR filter

$$\sum_{\ell=0}^N a_{\ell} y[n-\ell] = \sum_{k=0}^M b_k x[n-k]$$

- The  $a_{\ell}$ 's are called **feedback** coefficients, because they feed  $y[n]$  back into itself.

# Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation equation:

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$
$$Y(z) = X(z) + bz^{-1}X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + bz^{-1}}{1 - az^{-1}}$$

# The Pole and Zero of $H(z)$

- The pole,  $z = a$ , and zero,  $z = -b$ , are the values of  $z$  for which  $H(z) = \infty$  and  $H(z) = 0$ , respectively.
- But what does that mean? We know that for  $z = e^{j\omega}$ ,  $H(z)$  is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the pole and zero do not normally have unit magnitude.

- What it means is that:
  - When  $\omega = \angle(-b)$ , then  $|H(\omega)|$  is as close to a zero as it can possibly get, so at that that frequency,  $|H(\omega)|$  is as low as it can get.
  - When  $\omega = \angle a$ , then  $|H(\omega)|$  is as close to a pole as it can possibly get, so at that that frequency,  $|H(\omega)|$  is as high as it can get.

# Causality and Stability

- A filter is **causal** if and only if the output,  $y[n]$ , depends only on **current and past** values of the input,  $x[n], x[n - 1], x[n - 2], \dots$
- A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if  $|a| < 1$ .

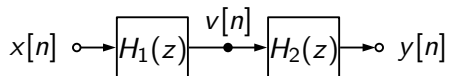
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# Series Combination

The series combination of two systems looks like this:



This means that

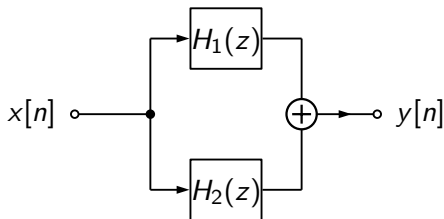
$$Y(z) = H_2(z)V(z) = H_2(z)H_1(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z)$$

# Parallel Combination

Parallel combination of two systems looks like this:



This means that

$$Y(z) = H_1(z)X(z) + H_2(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z)$$

# How to find the inverse Z transform

Any IIR filter  $H(z)$  can be written as...

- **denominator terms**, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

$$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n - k].$$

# Step #1: Numerator Terms

In general, if

$$G(z) = \frac{1}{A(z)}$$

for any polynomial  $A(z)$ , and

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{A(z)}$$

then

$$h[n] = b_0 g[n] + b_1 g[n-1] + \cdots + b_M g[n-M]$$

## Step #2: Partial Fraction Expansion

Partial fraction expansion works like this:

- 1 Factor  $A(z)$ :

$$G(z) = \frac{1}{\prod_{\ell=1}^N (1 - p_{\ell}z^{-1})}$$

- 2 Assume that  $G(z)$  is the result of a parallel system combination:

$$G(z) = \frac{C_1}{1 - p_1z^{-1}} + \frac{C_2}{1 - p_2z^{-1}} + \dots$$

- 3 Find the constants,  $C_{\ell}$ , that make the equation true. Such constants always exist, as long as none of the roots are repeated ( $p_k \neq p_{\ell}$  for  $k \neq \ell$ ).

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## How to Implement a Notch Filter

To implement a notch filter at frequency  $\omega_c$  radians/sample, with a bandwidth of  $-\ln(a)$  radians/sample, you implement the difference equation:

$$y[n] = x[n] - 2 \cos(\omega_c)x[n-1] + x[n-2] + 2a \cos(\omega_c)y[n-1] - a^2y[n-2]$$

which gives you the notch filter

$$H(z) = \frac{(1 - r_1z^{-1})(1 - r_1^*z^{-1})}{(1 - p_1z^{-1})(1 - p_1^*z^{-1})}$$

with the magnitude response:

$$|H(\omega)| = \begin{cases} 0 & \omega_c \\ \frac{1}{\sqrt{2}} & \omega_c \pm \ln(a) \\ \approx 1 & \omega < \omega_c - \ln(a) \text{ or } \omega > \omega_c + \ln(a) \end{cases}$$

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## A General Second-Order All-Pole Filter

Let's construct a general second-order all-pole filter (leaving out the zeros; they're easy to add later).

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

The difference equation that implements this filter is

$$Y(z) = X(z) + (p_1 + p_1^*)z^{-1}Y(z) - p_1 p_1^* z^{-2}Y(z)$$

Which converts to

$$y[n] = x[n] + 2\Re(p_1)y[n-1] - |p_1|^2 y[n-2]$$

# Understanding the Impulse Response of a Second-Order IIR

In order to **understand** the impulse response, maybe we should invent some more variables. Let's say that

$$p_1 = e^{-\sigma_1 + j\omega_1}, \quad p_1^* = e^{-\sigma_1 - j\omega_1}$$

where  $\sigma_1$  is the half-bandwidth of the pole, and  $\omega_1$  is its center frequency. The partial fraction expansion gave us the constant

$$C_1 = \frac{p_1}{p_1 - p_1^*} = \frac{p_1}{e^{-\sigma_1} (e^{j\omega_1} - e^{-j\omega_1})} = \frac{e^{j\omega_1}}{2j \sin(\omega_1)}$$

Therefore

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1)) u[n]$$

## Example: Ideal Resonator

Putting  $p_1 = e^{j\omega_1}$  into the general form, we find that the impulse response of this filter is

$$h[n] = \frac{1}{\sin(\omega_1)} \sin(\omega_1(n+1))u[n]$$

This is called an “ideal resonator” because it keeps ringing forever.

# Bandwidth

There are three frequencies that really matter:

- 1 Right at the pole, at  $\omega = \omega_1$ , we have

$$|e^{j\omega} - p_1| \approx \sigma_1$$

- 2 At  $\pm$  half a bandwidth,  $\omega = \omega_1 \pm \sigma_1$ , we have

$$|e^{j\omega} - p_1| \approx |-\sigma_1 \mp j\sigma_1| = \sigma_1\sqrt{2}$$

# 3dB Bandwidth

- The 3dB bandwidth of an all-pole filter is the width of the peak, measured at a level  $1/\sqrt{2}$  relative to its peak.
- $\sigma_1$  is half the bandwidth.

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- Notch Filters & Second-Order IIR