Lecture 32: Resonance

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ECE 401: Signal and Image Analysis

Outline

First-Order System

A causal IIR first-order system has the difference equation

$$y[n] = x[n] + ay[n-1]$$

Its system function is

$$H(z) = \frac{1}{1 - a_1 z^{-1}}$$

Its impulse response is

$$h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Second-Order System

A causal IIR second-order system has the difference equation

$$y[n] = x[n] + a_1 y[n-1] + a_2 y[n-2]$$

Its system function is

$$H(z) = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})},$$

where the relationship between the coefficients and the poles is $a_1 = p_1 + p_2$, $a_2 = -p_1 p_2$. Its impulse response is

$$h[n] = \begin{cases} C_1 p_1^n + C_2 p_2^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Impulse Response of a Second-Order All-Pole Filter

Suppose we write the pole as $p_1 = e^{-\sigma_1 + j\omega_1}$. Then we can write

$$H(z) = \frac{1}{1 - 2e^{-\sigma_1}\cos(\omega_1)z^{-1} + e^{-2\sigma_1}z^{-2}}$$

and

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1)) u[n]$$

Magnitude Response of a Second-Order All-Pole Filter

In the frequency response, there are three frequencies that really matter:

1. Right at the pole, at $\omega = \omega_1$, we have

$$|H(\omega_1)| \propto \frac{1}{\sigma_1}$$

2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|H(\omega_1 \pm \sigma_1)| = \frac{1}{\sqrt{2}}|H(\omega_1)|$$

Outline

Resonance

"Resonance describes the phenomenon of increased amplitude that occurs when the frequency of an applied periodic force (or a Fourier component of it) is equal or close to a natural frequency of the system on which it acts."
- https://en.wikipedia.org/wiki/Resonance

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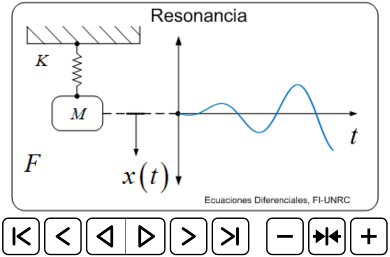

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Resonance in Discrete-Time Systems

In a discrete-time system, the "applied force" is $x[n]$. The "natural frequency" is ω_0, and σ is its damping:

$$y[n] = x[n] + 2e^{-\sigma}\cos(\omega_0)y[n-1] - e^{-2\sigma}y[n-2]$$

Resonance

Suppose "the frequency of the applied force is close to the natural frequency," i.e., $x[n] = e^{j\omega_A n}$:

$$y[n] = e^{j\omega_A n} + 2e^{-\sigma}\cos(\omega_0)y[n-1] - e^{-2\sigma_0}y[n-2] \qquad (1)$$

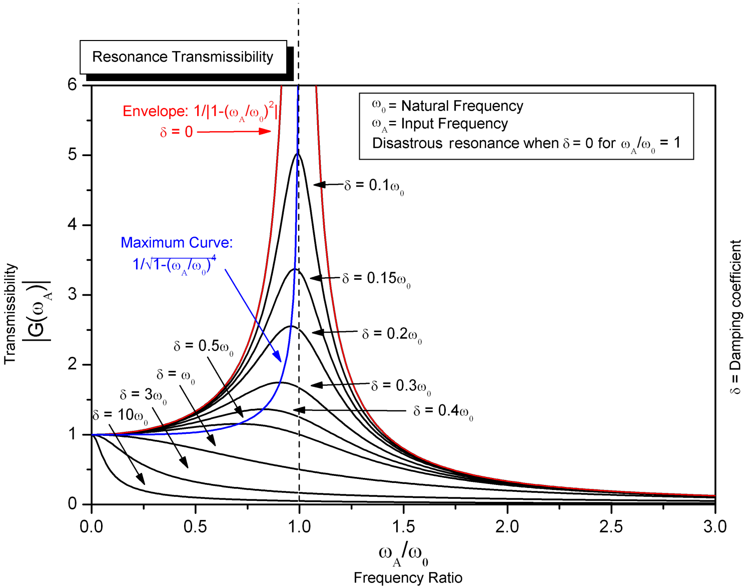
Since this is a linear, shift-invariant system, the output will be at the same frequency as the input:

$$y[n] = H(\omega_A)e^{j\omega_A n} \qquad (2)$$

Combining Eq. (1) and (2) gives us:

$$H(\omega_A) = \frac{1}{\left(1 - e^{-\sigma}e^{j(\omega_A-\omega_0)}\right)\left(1 - e^{-\sigma}e^{j(\omega_A+\omega_0)}\right)}$$

Resonance (https://commons.wikimedia.org/wiki/File:Resonance.PNG)



Outline

Natural Frequency

Suppose $x[n] = \delta[n]$ and $\sigma = 0$, then we have

$$y[n] = \delta[n] + 2\cos(\omega_0) y[n-1] - y[n-2]$$

- If the natural frequency is $\omega_0 = \frac{\pi}{3}$, then $y[n] = y[n-1] - y[n-2]$:

$$1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots$$

- If the natural frequency is $\omega_0 = \frac{\pi}{2}$, then $y[n] = -y[n-2]$:

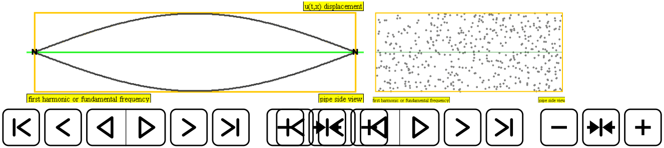
$$1, 0, -1, 0, 1, 0, -1, 0, 1, 0, \ldots$$

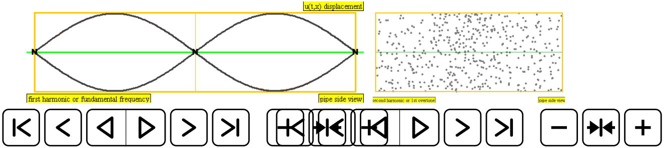
- If the natural frequency is $\omega_0 = \frac{2\pi}{3}$, then $y[n] = -y[n-1] - y[n-2]$:

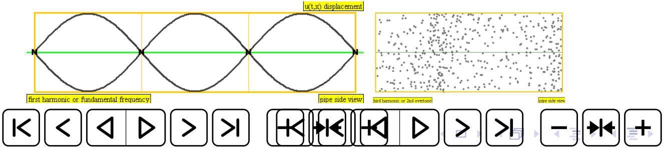
$$1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, \ldots$$

Natural Frequencies of Physical Systems

Natural frequencies of physical systems are determined by their size, shape, and materials. For example, the natural frequencies of a column of air, closed at both ends (a flute, or the vowel /u/) are $F_k = \frac{kc}{2L}$, where c is the speed of sound and L is the length:







Damped Impulse Response: Amplitude Decreases Toward Zero

Suppose $x[n] = \delta[n]$, $\omega_0 = \frac{\pi}{2}$, and $\sigma = -\ln(0.9) = 0.105$:

$$\begin{aligned} y[n] &= \delta[n] - (0.9)^2 y[n-2] \\ &= 1, 0, -(0.9)^2, 0, (0.9)^4, 0, -(0.9)^6, 0, \ldots \end{aligned}$$

"Applied Force:" Amplitude Increases toward $|H(\omega)|$

Suppose $x[n]$ is a cosine at the natural frequency:

$$\begin{aligned} y[n] &= \cos\left(\frac{\pi n}{2}\right) - (0.9)^2 y[n-2] \\ &= 1, 0, -1-(0.9)^2, 0, 1+(0.9)^2+(0.9)^4, \ldots \end{aligned}$$

Outline

Finding the Natural Frequency

Suppose we're given a system

$$y[n] = x[n] - by[n-1] - cy[n-2]$$

How can we find its resonant frequency?
Answer: use the quadratic formula!!

Finding the Natural Frequency

$$Y(z) = X(z) - bz^{-1}Y(z) - cz^{-2}Y(z)$$

$$\frac{Y(z)}{X(z)} = \frac{z^2}{z^2 + bz + c} = \frac{z^2}{(z - p_1)(z - p_2)}$$

So the poles are

$$p_1, p_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Overdamped System: $b^2 > 4c$

$$p_1, p_2 = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

Notice that if $b^2 > 4c$, then both p_1 and p_2 are real numbers! This is called an **overdamped system**. It is overdamped in the sense that it doesn't resonate, because b is too large. Instead of resonating, the impulse response is just the sum of two exponential decays:

$$h[n] = \begin{cases} C_1 p_1^n + C_2 p_2^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Underdamped System: $b^2 < 4c$

$$p_1, p_2 = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{b}{2} \pm j\frac{\sqrt{4c - b^2}}{2}$$

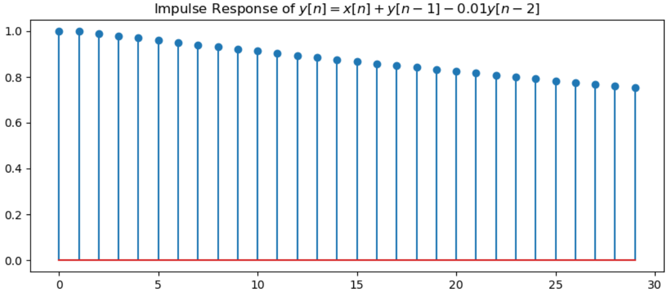
If $b^2 < 4c$, then both p_1 and p_2 are complex numbers, so the system resonates. This is called an **underdamped system**, and as we've seen, the impulse response is

$$h[n] = \begin{cases} \frac{1}{\sin(\omega_0)} e^{-\sigma n} \sin(\omega_0(n+1)) & n \ge 0 \\ 0 & n < 0 \end{cases}$$

where $-\sigma + j\omega_0 = \ln(p_1)$.

Comparison of Underdamped and Overdamped Systems

Suppose we set $y[n] = x[n] + y[n-1] - cy[n-2]$, and gradually increase c. Here's what happens:



Outline

Summary

$$y[n] = x[n] - by[n-1] - cy[n-2]$$

$$H(z) = \frac{1}{1 + bz^{-1} + cz^{-2}} = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-2})}$$

$$p_1, p_2 = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

Summary

- If $b^2 > 4c$, then the system is called **overdamped**. Its poles are both real-valued, and

$$h[n] = C_1 p_1^n u[n] + C_2 p_2^n u[n]$$

- If $b^2 < 4c$, then the system is called **underdamped** or **resonant**.
  - Its poles are complex conjugates, $p_2 = p_1^*$.
  - Its natural frequency is $\omega_0 = \Im\{\ln(p_1)\} = \angle p_1$.
  - Its bandwidth is $2\sigma = -2\Re\{\ln(p_1)\} = -2\Re\{\ln|p_1|\}$.

Outline

Written Examples

1. Consider the system

$$y[n] = x[n] + 1.4y[n-1] - 0.75y[n-2]$$

What is its natural frequency? What is its bandwidth?

2. Suppose you want to create a system with the following impulse response:

$$h[n] \propto (0.9)^n \sin\left(\frac{\pi}{6}(n+1)\right)$$

Find b and c so that $y[n] = x[n] - by[n-1] - cy[n-2]$.