Lecture 29: Block Diagrams and the Inverse Z Transform

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis

Outline

FIR and IIR Filters

- An autoregressive filter is also called **infinite impulse response (IIR)**, because $h[n]$ has infinite length.
- A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).

System Functions

A first-order autoregressive filter,

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$

has the impulse response and system function

$$h[n] = a^n u[n] + ba^{n-1}u[n-1] \leftrightarrow H(z) = \frac{1+bz^{-1}}{1-az^{-1}},$$

where a is called the **pole** of the filter, and $-b$ is called its **zero**.

Causality and Stability

- A filter is **causal** if and only if the output, $y[n]$, depends only an **current and past** values of the input, $x[n], x[n-1], x[n-2], \ldots$.
- A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if $|a| < 1$.

Outline

Why use block diagrams?

A first-order difference equation looks like

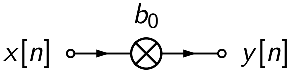
$$y[n] = b_0 x[n] + b_1 x[n-1] + a y[n-1]$$

- It's pretty easy to understand what computation is taking place in a first-order difference equation.
- As we get to higher-order systems, though, the equations for implementing them will be kind of complicated.
- In order to make the complicated equations very easy, we represent the equations using block diagrams.

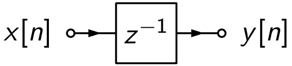
Elements of a block diagram

A block diagram has just three main element types:

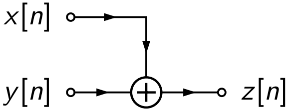
1. **Multiplier:** the following element means $y[n] = b_0 x[n]$:



2. **Unit Delay:** the following element means $y[n] = x[n-1]$ (i.e., $Y(z) = z^{-1}X(z)$):

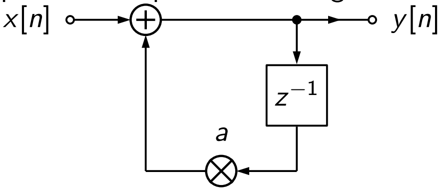


3. **Adder:** the following element means $z[n] = x[n] + y[n]$:



Example: Time Domain

Here's an example of a complete block diagram:



This block diagram is equivalent to the following equation:

$$y[n] = x[n] + ay[n-1]$$

Notice that we can read it, also, as

$$Y(z) = X(z) + az^{-1}Y(z) \quad \Rightarrow \quad H(z) = \frac{1}{1 - az^{-1}}$$

A Complete First-Order IIR Filter

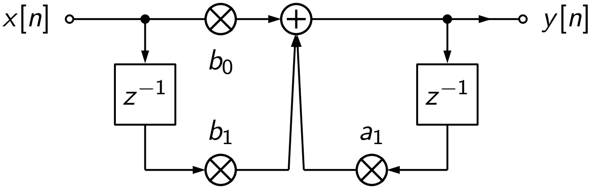
Now consider how we can represent a complete first-order IIR filter, including both the pole and the zero. Here it is in the z-domain:

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + a_1 z^{-1} Y(z).$$

When we implement it, we would write a line of python that does this:

$$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1],$$

which is exactly this block diagram:



Series and Parallel Combinations

Now let's talk about how to combine systems.

- **Series combination**: passing the signal through two systems **in series** is like multiplying the system functions:

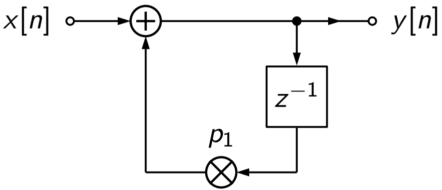
$$H(z) = H_2(z)H_1(z)$$

- **Parallel combination**: passing the signal through two systems in **parallel**, then adding the outputs, is like adding the system functions:

$$H(z) = H_1(z) + H_2(z)$$

One Block for Each System

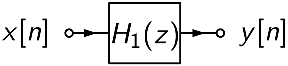
Suppose that one of the two systems, $H_1(z)$, looks like this:



and has the system function

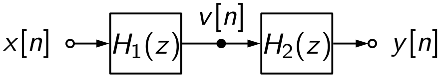
$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}$$

Let's represent the whole system using a single box:



Series Combination

The series combination, then, looks like this:



This means that

$$Y(z) = H_2(z)V(z) = H_2(z)H_1(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z)$$

Series Combination

The series combination, then, looks like this:

$$x[n] \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow y_2[n]$$

Suppose that we know each of the systems separately:

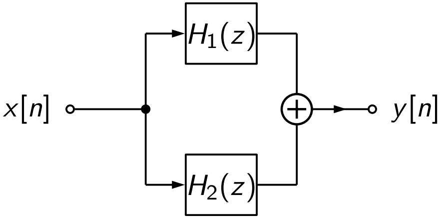
$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \qquad H_2(z) = \frac{1}{1 - p_2 z^{-1}}$$

Then, to get $H(z)$, we just have to multiply:

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{1}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}}$$

Parallel Combination

Parallel combination of two systems looks like this:



This means that

$$Y(z) = H_1(z)X(z) + H_2(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z)$$

Parallel Combination

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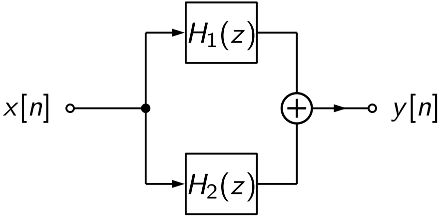


Suppose that we know each of the systems separately:

$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \qquad H_2(z) = \frac{1}{1 - p_2 z^{-1}}$$

Then, to get $H(z)$, we just have to add:

$$H(z) = \frac{1}{1 - p_1 z^{-1}} + \frac{1}{1 - p_2 z^{-1}} = \frac{2 - (p_1 + p_2) z^{-1}}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}}$$

Outline

Inverse Z transform

Suppose you know $H(z)$, and you want to find $h[n]$. How can you do that?

How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- **denominator terms**, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

$$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n-k].$$

Step #1: Numerator Terms

Consider one that you already know:

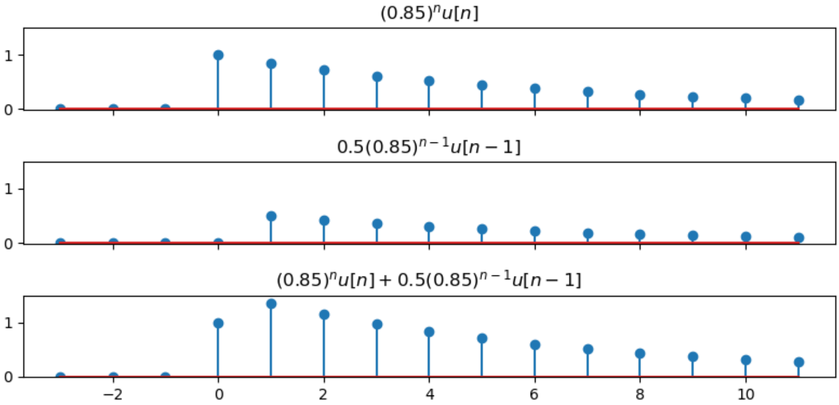
$$H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \left(\frac{1}{1 - az^{-1}}\right) + bz^{-1}\left(\frac{1}{1 - az^{-1}}\right)$$

and therefore

$$h[n] = \left(a^n u[n]\right) + b\left(a^{n-1} u[n-1]\right)$$

Step #1: Numerator Terms

So here is the inverse transform of $H(z) = \frac{1+0.5z^{-1}}{1-0.85z^{-1}}$:



Step #1: Numerator Terms

In general, if

$$G(z) = \frac{1}{A(z)}$$

for any polynomial $A(z)$, and

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{A(z)}$$

then

$$h[n] = b_0 g[n] + b_1 g[n-1] + \cdots + b_M g[n-M]$$

Step #2: Denominator Terms

Now we need to figure out the inverse transform of

$$G(z) = \frac{1}{A(z)}$$

We will solve this using a method called **partial fraction expansion.**

Step #2: Partial Fraction Expansion

Partial fraction expansion works like this:

1. Factor $A(z)$:
$$G(z) = \frac{1}{\prod_{\ell=1}^{N}\left(1 - p_\ell z^{-1}\right)}$$
2. Assume that $G(z)$ is the result of a parallel system combination:
$$G(z) = \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_2}{1 - p_2 z^{-1}} + \cdots$$
3. Find the constants, C_ℓ, that make the equation true. Such constants always exist, as long as none of the roots are repeated ($p_k \neq p_\ell$ for $k \neq \ell$).

Partial Fraction Expansion: Example

Step # 1: Factor it:

$$\frac{1}{1-1.2z^{-1}+0.72z^{-2}}=\frac{1}{\left(1-(0.6+j0.6)z^{-1}\right)\left(1-(0.6-j0.6)z^{-1}\right)}$$

Step #2: Express it as a sum:

$$\frac{1}{1-1.2z^{-1}+0.72z^{-2}}=\frac{C_1}{1-(0.6+j0.6)z^{-1}}+\frac{C_2}{1-(0.6-j0.6)z^{-1}}$$

Step #3: Find the constants. The algebra is annoying, but it turns out that:

$$C_1=\frac{1}{2}-j\frac{1}{2},\quad C_2=\frac{1}{2}+j\frac{1}{2}$$

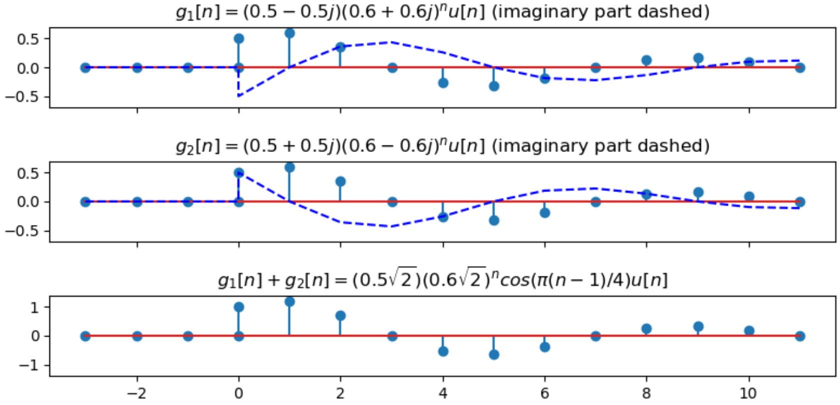
Partial Fraction Expansion: Example

The system function is:

$$\begin{aligned} G(z) &= \frac{1}{1-1.2z^{-1}+0.72z^{-2}} \\ &= \frac{0.5-0.5j}{1-(0.6+j0.6)z^{-1}} + \frac{0.5+0.5j}{1-(0.6-j0.6)z^{-1}} \end{aligned}$$

and therefore the impulse response is:

$$\begin{aligned} g[n] &= (0.5-0.5j)(0.6+0.6j)^n u[n] + (0.5+0.5j)(0.6-j0.6)^n u[n] \\ &= \left(0.5\sqrt{2}e^{-j\frac{\pi}{4}}\left(0.6\sqrt{2}e^{j\frac{\pi}{4}}\right)^n + 0.5\sqrt{2}e^{j\frac{\pi}{4}}\left(0.6\sqrt{2}e^{-j\frac{\pi}{4}}\right)^n\right) u[n] \\ &= \sqrt{2}(0.6\sqrt{2})^n \cos\left(\frac{\pi}{4}(n-1)\right) u[n] \end{aligned}$$


$g_1[n] = (0.5 - 0.5j)(0.6 + 0.6j)^n u[n]$ (imaginary part dashed)
0.5
0.0
−0.5
$g_2[n] = (0.5 + 0.5j)(0.6 - 0.6j)^n u[n]$ (imaginary part dashed)
0.5
0.0
−0.5
$g_1[n] + g_2[n] = (0.5\sqrt{2})(0.6\sqrt{2})^n cos(\pi(n-1)/4)u[n]$
1
0
−1
−2
0
2
4
6
8
10

How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- a **partial fraction expansion** into a sum of **denominator** terms, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

$$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n-k].$$

Outline

Summary: Block Diagrams

- A **block diagram** shows the delays, additions, and multiplications necessary to compute output from input.
- **Series combination**: passing the signal through two systems **in series** is like multiplying the system functions:

$$H(z) = H_2(z)H_1(z)$$

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Outline

Written Example

Find the inverse Z transform of

$$H(z) = \frac{1 - 0.7z^{-1}}{1 - 0.81z^{-2}}$$