Lecture 16: Cascaded LSI Systems

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2022

Outline

Review: Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

Frequency Response

- **Tones in → Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = G(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |G(\omega)| \cos(\omega n + \angle G(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|G(\omega)| \cos(\omega n + \theta + \angle G(\omega))$$

- where the **Frequency Response** is given by

$$G(\omega) = \sum_m g[m]e^{-j\omega m}$$

Review: Spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\,(x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Review: Fourier Series

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

Review: Discrete-Time Fourier Series

A signal that's periodic in discrete time also has a Fourier series. If the signal is periodic with a period of $N_0 = T_0 F_s$ samples, then its Fourier series is

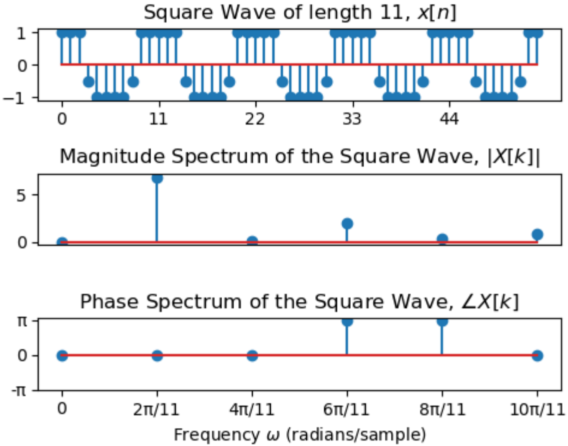
$$x[n] = \sum_{k=0}^{N_0-1} X_k e^{j2\pi kn/N_0} = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

and the Fourier analysis formula is

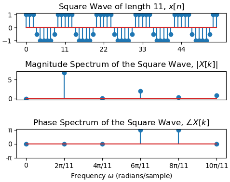
$$X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j2\pi kn/N_0}$$

Example: Spectrum of a Square Wave

For example, here's an even-symmetric ($x[n] = x[-n]$), zero-DC ($\sum_n x[n] = 0$), unit-amplitude ($\max_n |x[n]| = 1$) square wave, with a period of 11 samples:



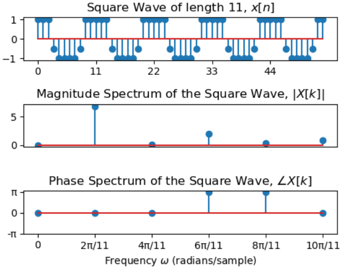
Spectrum of a Square Wave



The Fourier series coefficients of this square wave are

$$X_k = \begin{cases} \frac{2\pi}{k}(-1)^{\frac{|k|-1}{2}} & k \text{ odd}, \quad -5 \le k \le 5 \\ 0 & k \text{ even}, \quad -5 \le k \le 5 \end{cases}$$

More about the phase spectrum

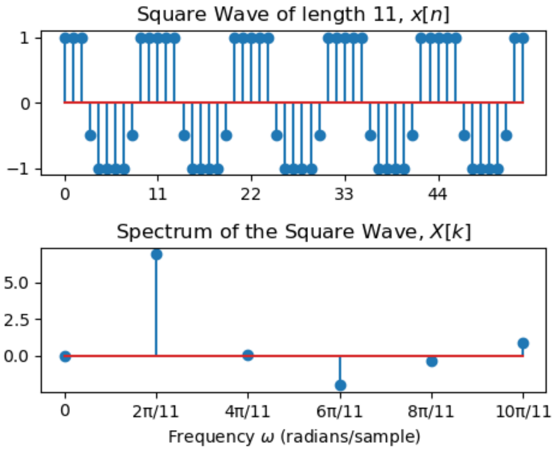


Notice that, for the phase spectrum of a square wave, the phase spectrum is either $\angle X[k] = 0$ or $\angle X[k] = \pi$. That means that the spectrum is real-valued, with no complex part:

- **Positive real:** $X[k] = |X[k]|$
- **Negative real:** $X[k] = -|X[k]| = |X[k]|e^{j\pi}$

More about the phase spectrum

Having discovered that the square wave has a real-valued $X[k]$, we could just plot $X[k]$ itself, instead of plotting its magnitude and phase:



Outline

Response of a Filter when the Input is Periodic

Now we're ready to ask this question:

What is the output of a filter when the input, $x[n]$, is periodic with period N_0?

Response of a Filter when the Input is Periodic

1. **Fourier Series:** If the input is periodic, then we can write it as

$$x[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

2. **Frequency Response:** If the input is $e^{j\omega n}$, then the output is

$$y[n] = H(\omega)e^{j\omega n}$$

3. **Linearity (of convolution, and of frequency response):** If the input is $x_1[n] + x_2[n]$, then the output is

$$y[n] = y_1[n] + y_2[n]$$

Response of a Filter when the Input is Periodic

Putting all those things together, if the input is

$$x[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

. . . then the output is

$$y[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k H(k\omega_0) e^{j2\pi kn/N_0}$$

. . . where $\omega_0 = \frac{2\pi}{N_0}$ is the fundamental frequency.

Outline

A Pure-Delay "Filter"

One thing we can do to a signal is to just delay it, by n_0 samples:

$$y[n] = x[n - n_0]$$

Even this very simple operation can be written as a convolution:

$$y[n] = g[n] * x[n]$$

where the "filter," $g[n]$, is just

$$g[n] = \delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$

Frequency Response of A Pure-Delay “Filter”

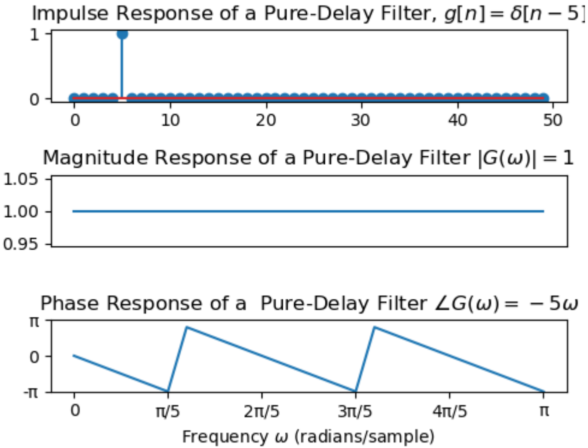
$$g[n] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$

The frequency response is

$$G(\omega) = \sum_m g[m] e^{-j\omega m} = e^{-j\omega n_0}$$

Impulse Response of A Pure-Delay "Filter"

Here is the impulse response of a pure-delay "filter" (and the magnitude and phase responses, which we'll talk about next).



Magnitude and Phase Response of A Pure-Delay "Filter"

$$G(\omega) = \sum_m g[m] e^{-j\omega m} = e^{-j\omega n_0}$$

Notice that the magnitude and phase response of this filter are

$$|G(\omega)| = 1$$
$$\angle G(\omega) = -\omega n_0$$

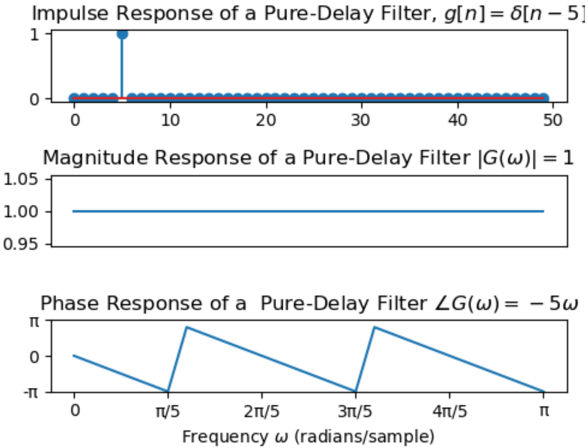
So, for example, if have an input of $x[n] = \cos(\omega n)$, the output would be

$$y[n] = |G(\omega)| \cos\left(\omega n + \angle G(\omega)\right) = \cos\left(\omega n - \omega n_0\right)$$

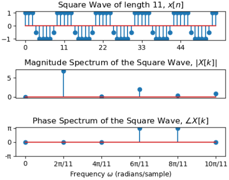
Outline

Magnitude and Phase Response of A Pure-Delay "Filter"

Here are the magnitude and phase response of the pure delay filter.



Spectrum of a Square Wave



Here are the Fourier series coefficients of a period-11, even-symmetric, unit-amplitude, zero-mean square wave:

$$X_k = \begin{cases} \frac{2\pi}{k}(-1)^{\frac{|k|-1}{2}} & k \text{ odd}, \quad -5 \le k \le 5 \\ 0 & k \text{ even}, \quad -5 \le k \le 5 \end{cases}$$

Response of a Filter when the Input is Periodic

And here's what happens when we pass a periodic signal through a filter $g[n]$:

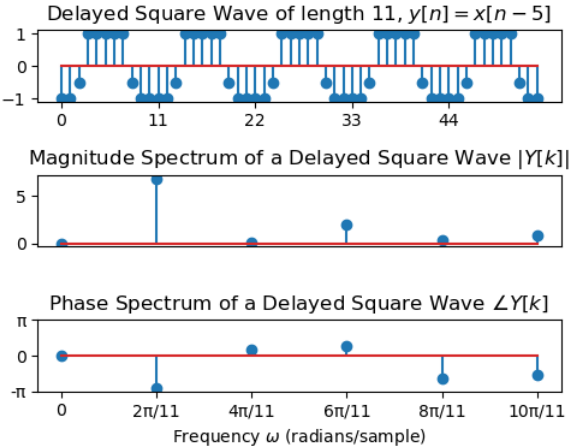
$$x[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

$$y[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k G(k\omega_0) e^{j2\pi kn/N_0}$$

...where $\omega_0 = \frac{2\pi}{N_0}$ is the fundamental frequency.

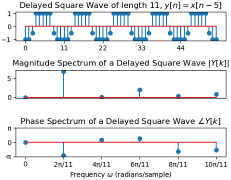
Spectrum: Delayed Square Wave

And here's the result. This is the square wave, after being delayed by the pure-delay filter:



You can see that magnitude's unchanged, but phase is changed.

Spectrum of a Delayed Square Wave



The Fourier series coefficients of a square wave, delayed by n_0 samples, are

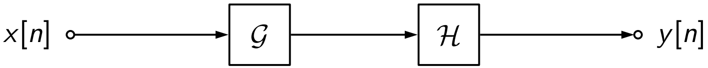
$$Y_k = \begin{cases} \frac{2\pi}{k}(-1)^{\frac{|k|-1}{2}} e^{-jk\omega_0 n_0} & k \text{ odd}, \quad -5 \le k \le 5 \\ 0 & k \text{ even}, \quad -5 \le k \le 5 \end{cases}$$

where $k\omega_0 = \frac{2\pi k}{N_0}$.

Outline

Cascaded LSI Systems

What happens if we pass the input through two LSI systems, in cascade?



Cascaded filters

Suppose I pass the signal through filter $g[n]$, then pass it through another filter, $h[n]$:

$$y[n] = h[n] * (g[n] * x[n]),$$

we get a signal $y[n]$ whose spectrum is:

$$Y[k] = H(k\omega_0)G(k\omega_0)X[k]$$

Convolution is Commutative

Notice that

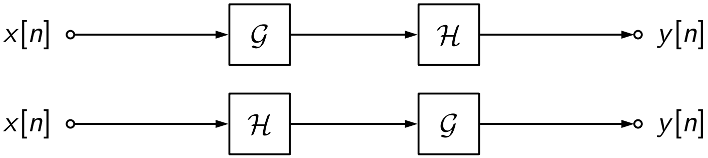
$$Y[k] = H(k\omega_0)G(k\omega_0)X[k] = G(k\omega_0)H(k\omega_0)X[k]$$

and therefore:

$$y[n] = h[n] * (g[n] * x[n]) = g[n] * (h[n] * x[n])$$

Convolution is Commutative

Since convolution is commutative, these two circuits compute exactly the same output:



Example: Differenced Square Wave

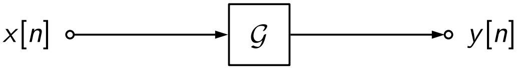
Suppose we define $x[n]$ to be an 11-sample square wave, $g[n]$ to be a delay, and $h[n]$ to be a first difference:

$$x[n] = \begin{cases} 1 & -2 \le n \le 2 \\ -\frac{1}{2} & n = \pm 3 \\ -1 & 4 \le n \le 7 \end{cases}$$

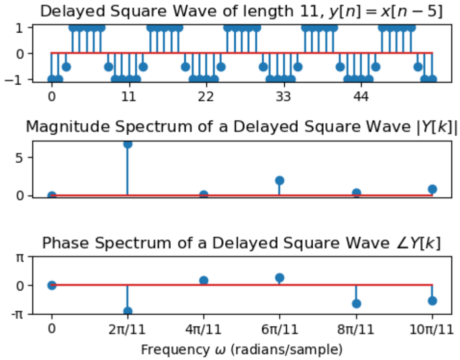
$$x[n] \xrightarrow{\mathcal{G}} z[n] = x[n-5]$$

$$z[n] \xrightarrow{\mathcal{H}} y[n] = z[n] - z[n-1]$$

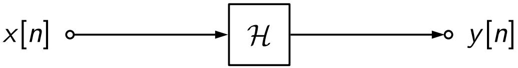
Delayed Square Wave



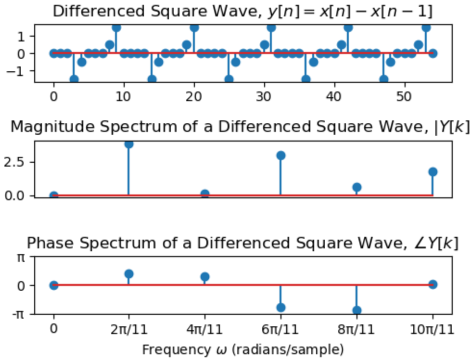
Here's what we get if we just **delay** the square wave:



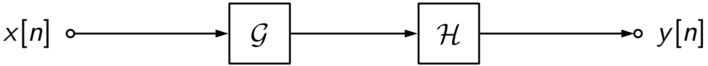
Differenced Square Wave



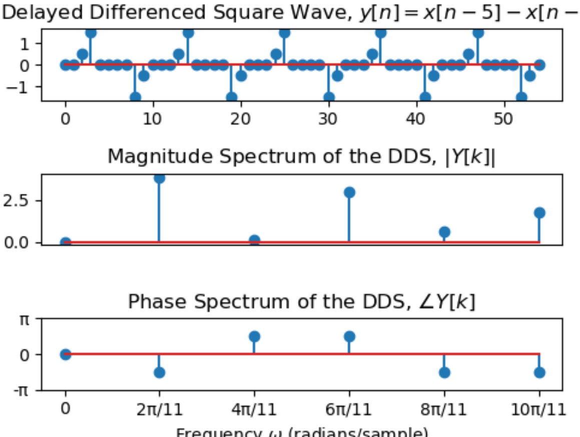
Here's what we get if we just **difference** the square wave:



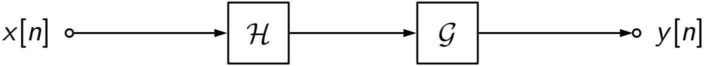
Example: Differenced Delayed Square Wave



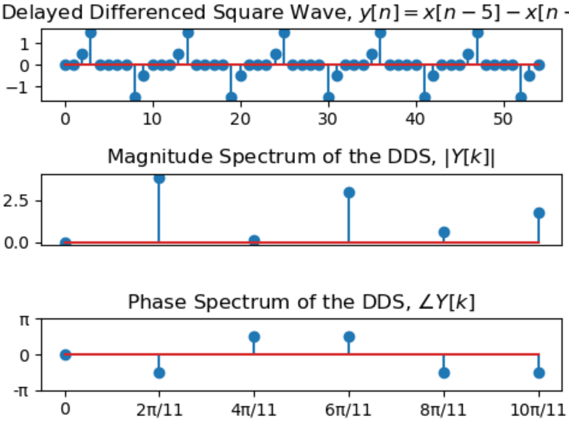
Here's what we get if we **delay** and then **difference** the square wave:



Example: Delayed Differenced Square Wave



Here's what we get if we **difference** and then **delay** the square wave (hint: it's exactly the same as the previous slide!!)



Magnitude and Phase of Cascaded Frequency Responses

In general, when you cascade two LSI systems, the magnitudes multiply:

$$|Y_k| = |H(\omega)||G(\omega)||X_k|,$$

but the phases add:

$$\angle Y_k = \angle H(\omega) + \angle G(\omega) + \angle X_k$$

That's because:

$$H(\omega)G(\omega) = |H(\omega)|e^{j\angle H(\omega)}|G(\omega)|e^{j\angle G(\omega)} = |H(\omega)||G(\omega)|e^{j(\angle H(\omega)+\angle G(\omega))}$$

Outline

Local Average Filters

Let's go back to the local averaging filter. I want to define two different types of local average: centered, and delayed.

- **Centered local average:** This one averages $\left(\frac{L-1}{2}\right)$ future samples, $\left(\frac{L-1}{2}\right)$ past samples, and $x[n]$:

$$y_c[n] = \frac{1}{L} \sum_{m=-\left(\frac{L-1}{2}\right)}^{\left(\frac{L-1}{2}\right)} x[n-m]$$

- **Delayed local average:** This one averages $x[n]$ and $L-1$ of its past samples:

$$y_d[n] = \frac{1}{L} \sum_{m=0}^{L-1} x[n-m]$$

Notice that $y_d[n] = y_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

Local Average Filters

We can write both of these as filters:

- **Centered local average:**

$$y_c[n] = f_c[n] * x[n]$$

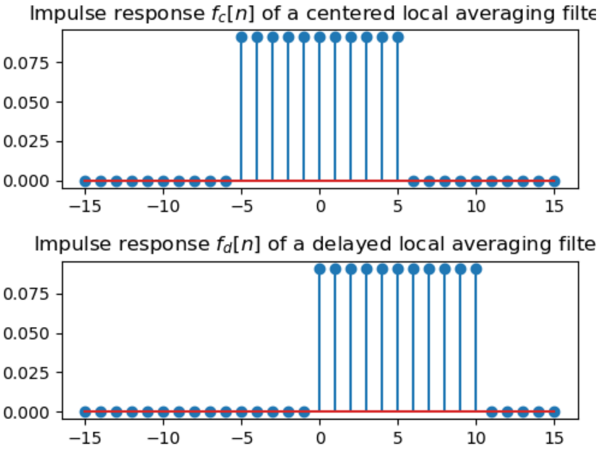
$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

- **Delayed local average:**

$$y_d[n] = f_d[n] * x[n]$$

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Local Average Filters



Notice that $f_d[n] = f_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

The relationship between centered local average and delayed local average

Suppose we define our pure delay filter,

$$g[n] = \delta\left[n - \frac{L-1}{2}\right] = \begin{cases} 1 & n = \frac{L-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Using $g[n]$, here are lots of different ways we can write the relationship between $y_d[n]$, $y_c[n]$, and $x[n]$:

$$\begin{aligned} y_d[n] &= f_d[n] * x[n] \\ y_c[n] &= f_c[n] * x[n] \\ y_d[n] &= g[n] * y_c[n] = g[n] * f_c[n] * x[n] \\ f_d[n] &= g[n] * f_c[n] \end{aligned}$$

The relationship between centered local average and delayed local average

Remember the frequency response of a pure delay filter:

$$G(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)}$$

We have not yet figured out what $F_c(\omega)$ and $F_d(\omega)$ are. But whatever they are, we know that

$$f_d[n] = g[n] * f_c[n]$$

and therefore

$$F_d(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)} F_c(\omega)$$

The frequency response of a local average filter

Let's find the frequency response of

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

The formula is

$$F_d(\omega) = \sum_m f[m]e^{-j\omega m},$$

so,

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

The frequency response of a local average filter

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

This is just a standard geometric series,

$$\sum_{m=0}^{L-1} a^m = \frac{1-a^L}{1-a},$$

so:

$$F_d(\omega) = \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right)$$

The frequency response of a local average filter

We now have an extremely useful transform pair:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right)$$

Let's attempt to convert that into polar form, so we can find magnitude and phase response. Notice that both the numerator and the denominator are subtractions of complex numbers, so we might be able to use $2j\sin(x) = e^{jx} - e^{-jx}$ for some x. Let's try:

$$\begin{aligned} \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right) &= \frac{1}{L}\frac{e^{-j\omega L/2}}{e^{-j\omega/2}}\left(\frac{e^{j\omega L/2}-e^{-j\omega L/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\ &= e^{-j\omega\left(\frac{L-1}{2}\right)}\frac{1}{L}\left(\frac{2j\sin(\omega L/2)}{2j\sin(\omega/2)}\right) \\ &= e^{-j\omega\left(\frac{L-1}{2}\right)}\frac{1}{L}\left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right) \end{aligned}$$

The frequency response of a local average filter

Now we have $F_d(\omega)$ in almost magnitude-phase form:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \left(\frac{\sin(\omega L/2)}{L\sin(\omega/2)}\right) e^{-j\omega\left(\frac{L-1}{2}\right)}$$

By the way, remember we discovered that

$$f_d[n] = g[n] * f_c[n] \quad \leftrightarrow \quad F_d(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)} F_c(\omega)$$

Notice anything?

Dirichlet form

The textbook calls this function the "Dirichlet form:"

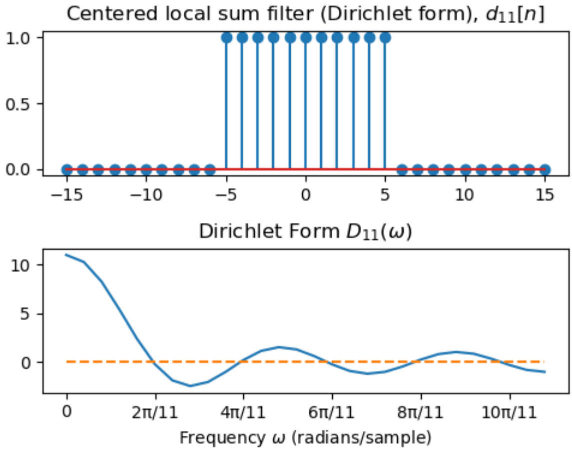
$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

That is, exactly, the frequency response of a centered local sum filter:

$$d_L[n] = \begin{cases} 1 & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Dirichlet form

Here's what it looks like:



Dirichlet form

Since every local averaging filter is based on Dirichlet form, it's worth spending some time to understand it better.

$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

- It's equal to zero every time $\omega L/2$ is a multiple of π. So

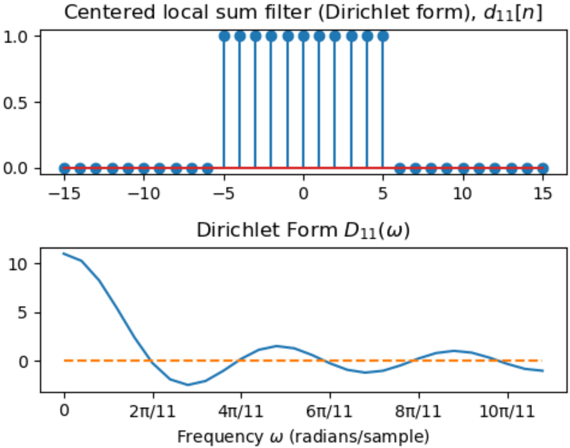
$$D_L\left(\frac{2\pi k}{L}\right) = 0 \text{ for all integers } k \text{ except } k = 0$$

- At $\omega = 0$, the value of $\frac{\sin(\omega L/2)}{\sin(\omega/2)}$ is undefined, but it's posssible to prove that $\lim_{\omega\to 0} D_L(\omega) = L$. To make life easy, we'll just define it that way:

$$\text{DEFINE:} \quad D_L(0) = L$$

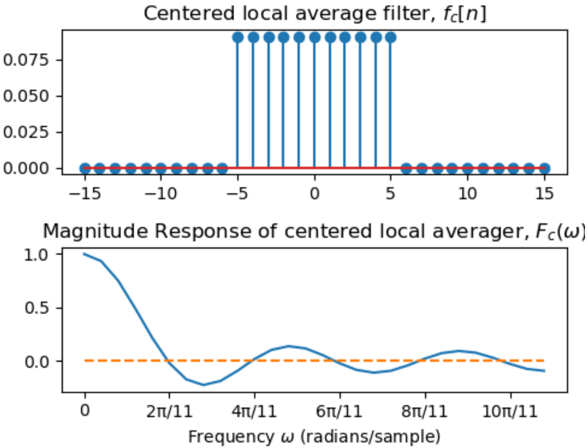
Dirichlet form

Here's what it looks like:



Local averaging filter

Here's what the centered local averaging filter looks like. Notice that it's just $1/L$ times the Dirichlet form:



Outline

Summary: Behavior of Systems in General

- **Periodic inputs:** If the input of an LSI system is periodic,

$$x[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

…then the output is

$$y[n] = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k H(k\omega_0) e^{j2\pi kn/N_0}$$

- **Cascaded LTI Systems** convolve their impulse responses, equivalently, they multiply their frequency responses:

$$y[n] = h[n] * g[n] * x[n], \quad Y_k = H(k\omega_0)G(k\omega_0)X_k$$

Summary: Types of LSI Systems

- The **Pure Delay Filter** has $|G(\omega)| = 1$, $\angle G(\omega) = -\omega n_0$:

$$g[m] = \delta[n - n_0] \quad \leftrightarrow \quad G(\omega) = e^{-j\omega n_0}$$

- The **Centered Local Averaging Filter** is $1/L$ times the Dirichlet form:

$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_c(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)}$$

- The **Delayed Local Averaging Filter** is $f_c[n]$, delayed by half of its length:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)} e^{-j\omega\left(\frac{L-1}{2}\right)}$$