Lecture 2: Sines, Cosines and Complex Exponentials

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2022

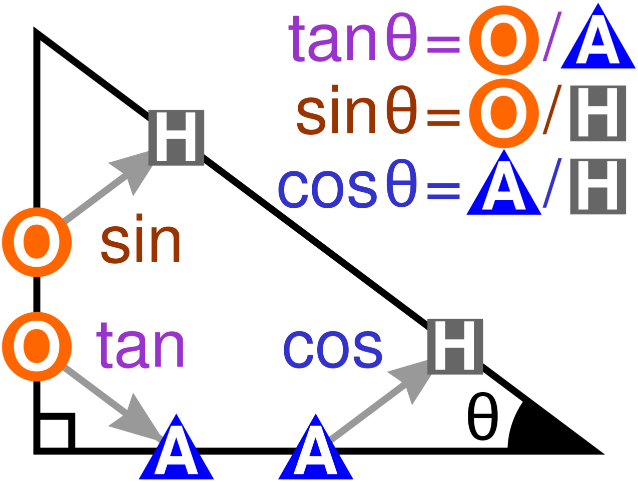
Outline

SOHCAHTOA

Sine and Cosine functions were invented to describe the sides of a right triangle:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$
$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$
$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

SOHCAHTOA



Sines, Cosines, and Circles

Imagine an ant walking counter-clockwise around a circle of radius A. Suppose the ant walks all the way around the circle once every T seconds.

- The ant's horizontal position at time t, $x(t)$, is given by

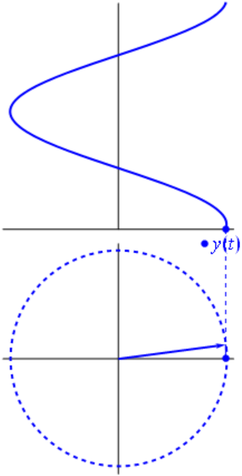
$$x(t) = A\cos\left(\frac{2\pi t}{T}\right)$$

- The ant's vertical position, $y(t)$, is given by

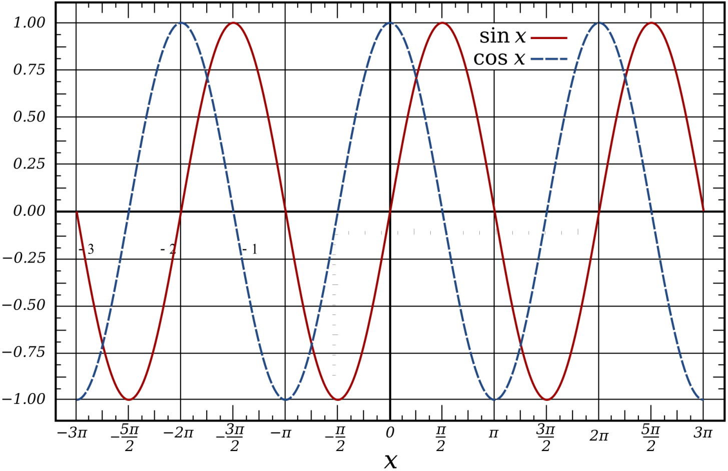
$$y(t) = A\sin\left(\frac{2\pi t}{T}\right)$$

Sines, Cosines, and Circles

by Gonfer, CC-SA 3.0, https://commons.wikimedia.org/wiki/File:Unfasor.gif



$x(t)$ and $y(t)$



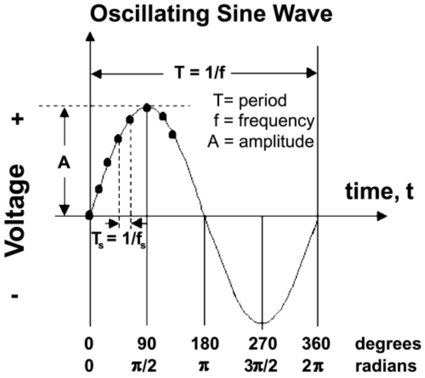
By Inductiveload, public domain image 2008,

`https://commons.wikimedia.org/wiki/File:Sine_and_Cosine.svg`

Period and Frequency

The period of a cosine, T, is the time required for one complete cycle. The frequency, $f = 1/T$, is the number of cycles per second. This picture shows

$$y(t) = A\sin\left(\frac{2\pi t}{T}\right) = A\sin(2\pi f t)$$



Pure Tones

In music or audiometry, a "pure tone" at frequency f is an acoustic signal, $p(t)$, given by

$$p(t) = A\cos(2\pi f t + \theta)$$

for any amplitude A and phase θ.

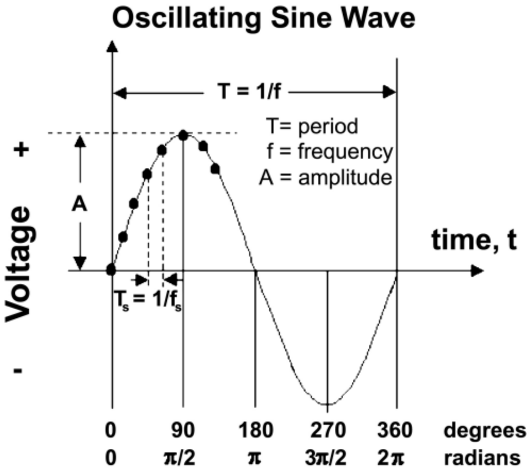
Pure Tone Demo

Phase, Distance, and Time

Remember the ant on the circle. The circle has a radius of A (say, A centimeters).

- When the ant has walked a distance of A centimeters around the outside of the circle, then it has moved to an angle of 1 radian.
- When the ant walks all the way around the circle, it has walked $2\pi A$ centimeters, which is 2π radians.

Phase, Distance, and Time



National Institute of Standards and Technology, public domain image 2010 https://www.nist.gov/pml/time-and-frequency-division/popular-links/time-frequency-z/time-and-frequency-z-p

Phase Shift

Where did the ant start?

- If the ant starts at an angle of θ, and continues walking counter-clockwise at f cycles/second, then

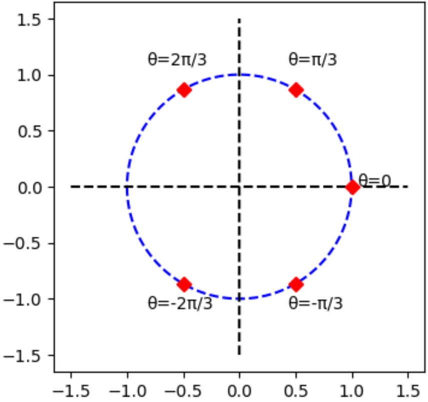
$$x(t) = A\cos\left(\frac{2\pi t}{T} + \theta\right)$$

- This is exactly the same as if it started walking from phase 0 at time $-\tau = -\frac{\theta}{2\pi f}$:

$$x(t) = A\cos\left(\frac{2\pi}{T}(t+\tau)\right), \quad \tau = \frac{T\theta}{2\pi} = \frac{\theta}{2\pi f}$$

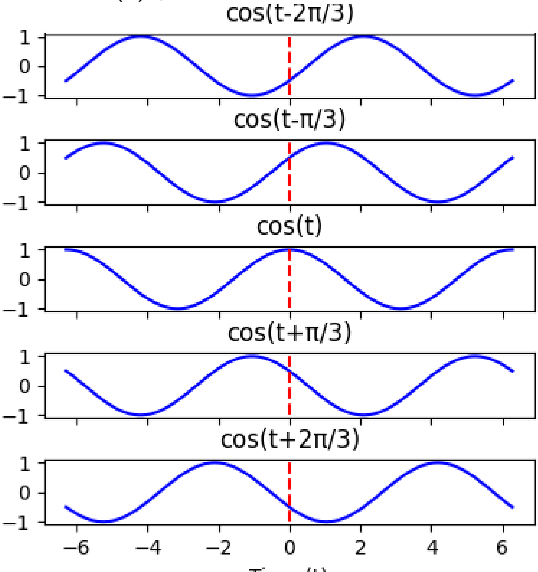
Phase Shift

Where did the ant start?



Phase Shift

What is the ant's $x(t)$ position, based on where it started?



Outline

Beat tones

When two pure tones at similar frequencies are added together, you hear the two tones “beating” against each other.

Beat tones demo

Beat tones and Trigonometric identities

Beat tones can be explained using this trigonometric identity:

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

Let's do the following variable substitution:

$$\begin{aligned} a+b &= 2\pi f_1 t \\ a-b &= 2\pi f_2 t \\ a &= 2\pi f_{ave} t \\ b &= 2\pi f_{beat} t \end{aligned}$$

where $f_{ave} = \frac{f_1+f_2}{2}$, and $f_{beat} = \frac{f_1-f_2}{2}$.

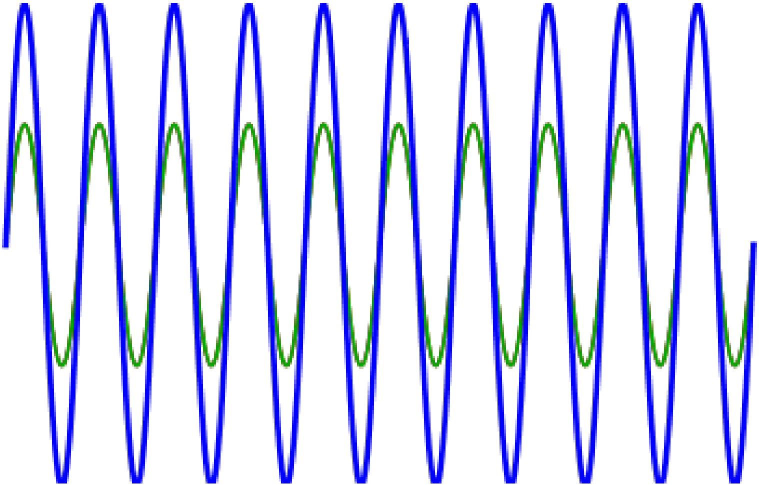
Beat tones and Trigonometric identities

Re-writing the trigonometric identity, we get:

$$\frac{1}{2}\cos(2\pi f_1 t) + \frac{1}{2}\cos(2\pi f_2 t) = \cos(2\pi f_{beat} t)\cos(2\pi f_{ave} t)$$

So when we play two tones together, $f_1 = 110$Hz and $f_2 = 104$Hz, it sounds like we're playing a single tone at $f_{ave} = 107$Hz, multiplied by a beat frequency $f_{beat} = 3$ (double beats)/second.

Beat tones



More complex beat tones

What happens if we add together, say, three tones?

$$\cos(2\pi 107t) + \cos(2\pi 110t) + \cos(2\pi 104t) = ???$$

For this, and other more complicated operations, it is much, much easier to work with complex exponentials, instead of cosines.

Outline

Euler's Identity

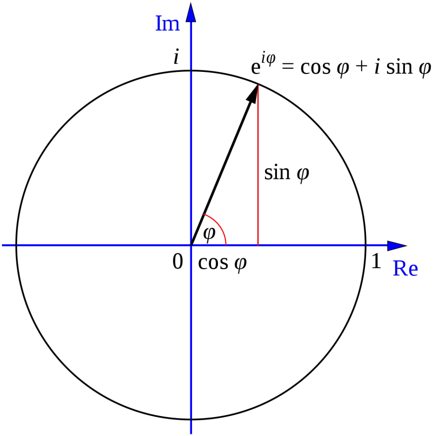
Euler asked: "What is $e^{j\theta}$?" He used the exponential summation:

$$e^x = 1 + x + \frac{1}{2}x^2 + \ldots \frac{1}{n!}x^n + \ldots$$

to show that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Euler's formula



By Gunther, CC-SA 3.0, `https://commons.wikimedia.org/wiki/File:Euler%27s_formula.svg`

Complex conjugates

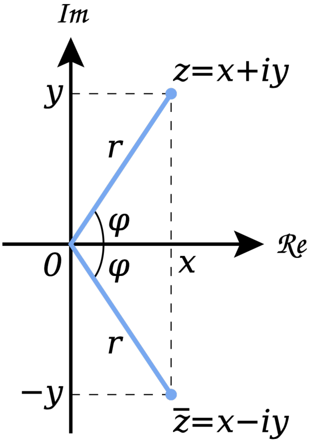
The polar form of a complex number is $z = re^{j\theta}$,

$$z = re^{j\theta} = r\cos\theta + jr\sin\theta$$

The complex conjugate is defined to be the mirror image of z, mirrored through the real axis:

$$z^* = re^{-j\theta} = r\cos\theta - jr\sin\theta$$

Complex conjugate



Real part of a complex number

If we know z and z^*,

$$z = re^{j\theta} = r\cos\theta + jr\sin\theta$$
$$z^* = re^{-j\theta} = r\cos\theta - jr\sin\theta$$

Then we can get the real part of z back again as

$$\Re\{z\} = \frac{1}{2}(z + z^*)$$

Why complex exponentials are better than cosines

Suppose we want to add together a lot of phase shifted, scaled cosines, all at the same frequency:

$$x(t) = A\cos(2\pi ft + \theta) + B\cos(2\pi ft + \phi) + C\cos(2\pi ft + \psi)$$

What is $x(t)$?

Why complex exponentials are better than cosines

We can simplify this problem by finding the **phasor representation** of the tones (I'll give you a formal definition of "phasor" in a few slides):

$$A\cos(2\pi ft+\theta) = \Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}$$

$$B\cos(2\pi ft+\phi) = \Re\left\{Be^{j\phi}e^{j2\pi ft}\right\}$$

$$C\cos(2\pi ft+\psi) = \Re\left\{Ce^{j\theta}e^{j2\psi ft}\right\}$$

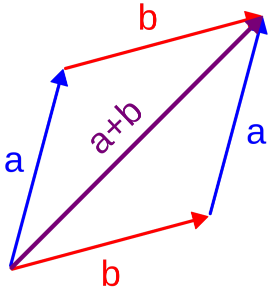
So

$$x(t) = \Re\left\{\left(Ae^{j\theta}+Be^{j\phi}+Ce^{j\psi}\right)e^{j2\pi ft}\right\}$$

Why complex exponentials are better than cosines

We add complex numbers by (1) adding their real parts, and (2) adding their imaginary parts:

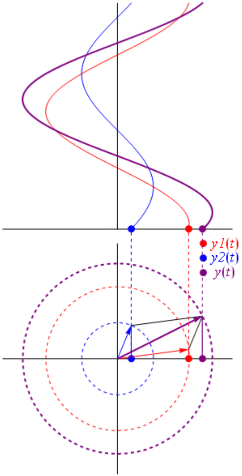
$$A e^{j\theta} + B e^{j\phi} + C e^{j\psi} = (A\cos\theta + B\cos\phi + C\cos\psi) + j(A\sin\theta + B\sin\phi + C\sin\psi)$$



By Booyabazooka, public domain image 2009,

https://commons.wikimedia.org/wiki/File:Vector_Addition.svg

Adding phasors



Why complex exponentials are better than cosines

Suppose we want to add together a lot of phase shifted, scaled cosines, all at the same frequency:

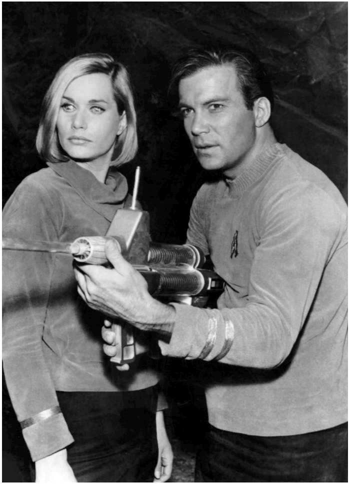
$$x(t) = A\cos(2\pi ft + \theta) + B\cos(2\pi ft + \phi) + C\cos(2\pi ft + \psi)$$

Here's the fastest way to do that:

1. Convert all the tones to their phasors, $a = Ae^{j\theta}$, $b = Be^{j\phi}$, and $c = Ce^{j\psi}$.
2. Add the phasors: $x = a + b + c$.
3. Take the real part:

$$x(t) = \Re\left\{xe^{j2\pi ft}\right\}$$

BTW, What is a “phaser”?



By McFadden, Strauss Eddy & Irwin for Desilu Productions, public domain image 1966,

https://commons.wikimedia.org/wiki/File:William_Shatner_Sally_Kellerman_Star_Trek_1966.JPG

BTW, What is a ~~"phaser"~~ "phasor"?

Wikipedia has the following definition, which is the best I've ever seen:

- The function $Ae^{j(\omega t+\theta)}$ is called the **analytic representation** of $A\cos(\omega t+\theta)$.
- It is sometimes convenient to refer to the entire function as a **phasor**. But the term **phasor** usually implies just the static vector $Ae^{j\theta}$.

In other words, the "phasor" can mean either $Ae^{j(\omega t+\theta)}$ or just $Ae^{j\theta}$. If you're asked for the phasor representation of some cosine, either answer is correct.

Some phasor demos from the textbook

Here are some phasor demos, provided with the textbook.

- **One rotating phasor demo**: This shows how the cosine, $\cos(2\pi ft+\theta)$, is the real part of the phasor $e^{j(2\pi ft+\theta)}$.
- **Positive and Negative Frequency Phasors**: This shows how you can get the real part of a phasor by adding its complex conjugate (its "negative frequency phasor"):

$$\cos(2\pi ft+\theta)=\frac{1}{2}e^{j(2\pi ft+\theta)}+\frac{1}{2}e^{-j(2\pi ft+\theta)}$$

Outline

Summary

- Cosines and Sines:

$$A\cos\left(\frac{2\pi t}{T}+\theta\right) = A\cos\left(2\pi f(t+\tau)\right)$$

- Beat Tones:

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

- Phasors:
  1. Convert all the tones to their phasors, $a = Ae^{j\theta}$, $b = Be^{j\phi}$, and $c = Ce^{j\psi}$.
  2. Add the phasors: $x = a + b + c$.
  3. Take the real part:

$$x(t) = \Re\left\{xe^{j2\pi ft}\right\}$$