UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
Fall 2022

# EXAM 3

Wednesday, December 14, 2022, 7:00-10:00pm

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 7:00pm; you will need to photograph and upload your answers by exactly 10:00pm.
- There will be a total of 200 points in the exam (9 problems). Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

netid: ______________________________

**Phasors**

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

**Fourier Series**

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$
$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

**Sampling and Interpolation:**

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$
$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$
$$z_a=\begin{cases} z & f \bmod F_s < -f \bmod F_s\\ z^* & f \bmod F_s > -f \bmod F_s\end{cases}$$
$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

**Convolution**

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

**Frequency Response and DTFT**

$$H(\omega)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$
$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(\omega)e^{j\omega n}d\omega$$
$$h[n]*\cos(\omega n)=|H(\omega)|\cos\left(\omega n+\angle H(\omega)\right)$$

**Rectangular & Hamming Windows; Ideal LPF**

$$w_R[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{otherwise}\end{cases}\leftrightarrow W_R(\omega)=e^{-\frac{j\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$
$$w_H[n]=0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right)w_R[n]\leftrightarrow W_H(\omega)=0.54W_R(\omega)-0.23W_R\left(\omega-\frac{2\pi}{N-1}\right)-0.23W_R\left(\omega+\frac{2\pi}{N-1}\right)$$
$$h_{\text{ideal}}[n]=\frac{\omega_c}{\pi}\text{sinc}(\omega_c n)\leftrightarrow H_{\text{ideal}}(\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & \text{otherwise}\end{cases}$$

**Discrete Fourier Transform**

$$\textbf{Analysis:}\ \ X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}$$
$$\textbf{Synthesis:}\ \ x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N}$$

**Z Transform Pairs**

$$
\begin{aligned}
b_k z^{-k} &\leftrightarrow b_k \delta[n-k] \\
\frac{1}{1-az^{-1}} &\leftrightarrow a^n u[n] \\
\frac{1}{(1-e^{-\sigma_1 - \jmath\omega_1} z^{-1})(1-e^{-\sigma_1 + \jmath\omega_1} z^{-1})} &\leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1 (n+1)\right) u[n]
\end{aligned}
$$

1. (25 points) Suppose we have a continuous-time signal $x(t)$, given by

$$x(t) = (3+2j)e^{-j12\pi t/T_0} + (5-j)e^{-j6\pi t/T_0} + (5+j)e^{j6\pi t/T_0} + (3-2j)e^{j12\pi t/T_0}$$

(a) Let's multiply $x(t)$ by the cosine of $4\pi t/T_0$, and integrate over one period:

$$A = \int_0^{T_0} x(t)\cos\left(\frac{4\pi t}{T_0}\right) dt$$

What is $A$?

(b) Suppose we multiply, instead, by the cosine of $6\pi t/T_0$, and integrate over one period:

$$B = \int_0^{T_0} x(t)\cos\left(\frac{6\pi t}{T_0}\right) dt$$

What is $B$?

(c) Continuing with the same $x(t)$: suppose we sample $x(t)$ with a sampling period of $T = T_0/8$, thus

$$y[n] = x(t)|_{t=nT}$$

The resulting $y[n]$ is periodic with a period of 8 samples, and with a discrete-time Fourier series of

$$y[n] = \sum_{k=-3}^{3} Y_k e^{j\frac{2\pi kn}{8}}$$

Find the values of $Y_k$, for $-3 \leq k \leq 3$.

2. (20 points) Consider a linear shift-invariant system with the following impulse response:

$$h[n] = \begin{cases} (0.9)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(a) Is this system stable? Why or why not?

(b) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Use convolution to find $y[n]$. You may find it useful to know that $\sum_{n=0}^{L-1} a^n = \frac{1-a^L}{1-a}$.

3. (25 points) A linear shift-invariant system has the following frequency response:

$$H(\omega) = e^{-5j\omega}\left(1 + \frac{1}{2}\cos\omega\right)$$

(a) Suppose that $y[n] = h[n] * x[n]$, where

$$x[n] = 3\sin\left(\frac{\pi n}{6}\right)$$

What is $y[n]$?

(b) What is the impulse response, $h[n]$, of this system?

4. (20 points) Suppose you want an FIR bandpass filter with a length of $N = 1024$, with cutoff frequencies $\omega_{\mathrm{LO}} = 0.46\pi$ and $\omega_{\mathrm{HI}} = 0.48\pi$, and with little stopband ripple. Find an impulse response $h[n]$ that meets these requirements.

5. (25 points) Consider the signal

$$x[n] = \cos\left(\frac{\pi n}{100}\right)$$

(a) What is the 128-sample DFT, $X[k]$, of this signal? Be sure to consider windowing effects.

(b) Suppose you want to construct the following periodic signal:

$$y[n] = \cos\left(\frac{\pi(n - 128\ell)}{100}\right), \quad 128\ell \leq n < 128(\ell + 1), \quad \text{for all } \ell$$

You can create this signal using the following Fourier series:

$$y[n] = \sum_{k=0}^{127} Y_k e^{j\frac{2\pi kn}{128}}$$

Notice that there is a relationships between $Y_k$ and the DFT coefficients, $X[k]$, that you computed in part (a) of this problem. Find $Y_k$ in terms of $X[k]$.

6. (25 points) Suppose you have a very long signal, $x[n]$, that you want to filter to compute $y[n] = h[n] * x[n]$. This can be done using the following sequence of steps:

$$H[k] = \sum_{n=0}^{N-1} h[n] e^{-j\frac{2\pi kn}{N}} \tag{1}$$

$$x_\ell[n] = \begin{cases} x[n+\ell M] & 0 \le n \le M-1, \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$X_\ell[k] = \sum_{n=0}^{N-1} x_\ell[n] e^{-j\frac{2\pi kn}{N}} \tag{3}$$

$$Y_\ell[k] = H[k] X_\ell[k] \tag{4}$$

$$y_\ell[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y_\ell[k] e^{j\frac{2\pi kn}{N}} \tag{5}$$

$$y[n] = \sum_{\ell=-\infty}^{\infty} y_\ell[n - \ell M] \tag{6}$$

(a) Suppose $h[n]$ is 129 samples long. Find values of $M$ and $N$ such that the algorithm in Eqs. (1) through (6) gives the same result as $y[n] = h[n] * x[n]$. There are many different correct answers; you only need to find one correct answer.

(b) Suppose $x[n] = \cos(0.08\pi n)$, and $h[n] = \delta[n - 126]$. Find $x_\ell[n]$ and $y_\ell[n]$ in terms of $\ell$, $M$, and $n$.

7. (20 points) Suppose that

$$H(z) = \frac{1}{1 - 0.9e^{j\pi/6}z^{-1}} + \frac{1}{1 - 0.9e^{-j\pi/6}z^{-1}}$$

Find the pole(s) and zero(s) of $H(z)$.

8. (20 points) Consider a notch filter with zeros at $r_1 = e^{j0.47\pi}$ and $r_2 = e^{-j0.47\pi}$, and with poles at $p_1 = 0.999e^{j0.47\pi}$ and $p_2 = 0.999e^{-j0.47\pi}$.

   (a) What is the 3dB bandwidth of the notch, expressed in radians per sample?

   (b) This filter can be implemented as

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

   Find $b_1$, $b_2$, $a_1$, and $a_2$.

9. (20 points) A particular bell has a resonance at 440Hz, with a decay time of half a second, and another resonance at 1320Hz, with a decay time of 3 seconds. Find a filter, $H(z)$, whose impulse response sounds like the impulse response of this bell if played through a D/A at $F_s = 10,000$ samples per second.