EXAM 3

Wednesday, December 14, 2022, 7:00-10:00pm

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression \( e^{-5 \cos(3)} \) is a MUCH better answer than \(-0.00667\).
- If you’re taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 7:00pm; you will need to photograph and upload your answers by exactly 10:00pm.
- There will be a total of 200 points in the exam (9 problems). Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.
Phasors

\[ A \cos(2\pi ft + \theta) = \Re \{ A e^{j\theta} e^{j2\pi ft} \} = \frac{1}{2} e^{-j\theta} e^{-j2\pi ft} + \frac{1}{2} e^{j\theta} e^{j2\pi ft} \]

Fourier Series

**Analysis:**
\[ X_k = \frac{1}{T_0} \int_{0}^{T_0} x(t) e^{-j2\pi kt/T_0} dt \]

**Synthesis:**
\[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]

Sampling and Interpolation:
\[ x[n] = x \left( t = \frac{n}{F_s} \right) \]
\[ f_a = \min \left( f \mod F_s, -f \mod F_s \right) \]
\[ z_a = \begin{cases} 
  z & f \mod F_s < -f \mod F_s \\
  z^* & f \mod F_s > -f \mod F_s 
\end{cases} \]
\[ y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) \]

Convolution
\[ h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m] x[n - m] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]

Frequency Response and DTFT
\[ H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \]
\[ h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega \]
\[ h[n] \cos(\omega n) = |H(\omega)| \cos (\omega n + \angle H(\omega)) \]

Rectangular & Hamming Windows; Ideal LPF
\[ w_R[n] = \begin{cases} 
  1 & 0 \leq n \leq N-1 \\
  0 & \text{otherwise}
\end{cases} \leftrightarrow W_R(\omega) = e^{-j\omega(N-1)/2} \sin(\omega N/2) / (\omega/2) \]
\[ w_H[n] = 0.54 - 0.46 \cos \left( \frac{2\pi n}{N-1} \right) \leftrightarrow W_H(\omega) = 0.54 W_R(\omega) - 0.23 W_R \left( \omega - \frac{2\pi}{N-1} \right) - 0.23 W_R \left( \omega + \frac{2\pi}{N-1} \right) \]
\[ h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \sin(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 
  1 & |\omega| < \omega_c \\
  0 & \text{otherwise}
\end{cases} \]

Discrete Fourier Transform

**Analysis:**
\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \]

**Synthesis:**
\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \]
Z Transform Pairs

$$b_k z^{-k} \leftrightarrow b_k \delta[n - k]$$

$$\frac{1}{1 - az^{-1}} \leftrightarrow a^n u[n]$$

$$\frac{1}{(1 - e^{-\sigma_1 - j\omega_1 z^{-1}})(1 - e^{-\sigma_1 + j\omega_1 z^{-1}})} \leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin (\omega_1 (n + 1)) u[n]$$
1. (25 points) Suppose we have a continuous-time signal $x(t)$, given by

$$x(t) = (3 + 2j)e^{-j12\pi t/T_0} + (5 - j)e^{-j6\pi t/T_0} + (5 + j)e^{j6\pi t/T_0} + (3 - 2j)e^{j12\pi t/T_0}$$

(a) Let’s multiply $x(t)$ by the cosine of $4\pi t/T_0$, and integrate over one period:

$$A = \int_0^{T_0} x(t) \cos \left(\frac{4\pi t}{T_0}\right) dt$$

What is $A$?

(b) Suppose we multiply, instead, by the cosine of $6\pi t/T_0$, and integrate over one period:

$$B = \int_0^{T_0} x(t) \cos \left(\frac{6\pi t}{T_0}\right) dt$$

What is $B$?
(c) Continuing with the same $x(t)$: suppose we sample $x(t)$ with a sampling period of $T = T_0/8$, thus

$$y[n] = x(t)|_{t=nT}$$

The resulting $y[n]$ is periodic with a period of 8 samples, and with a discrete-time Fourier series of

$$y[n] = \sum_{k=-3}^{3} Y_k e^{j2\pi kn/8}$$

Find the values of $Y_k$, for $-3 \leq k \leq 3$. 
2. (20 points) Consider a linear shift-invariant system with the following impulse response:

\[ h[n] = \begin{cases} 
(0.9)^n & n \geq 0 \\
0 & n < 0 
\end{cases} \]

(a) Is this system stable? Why or why not?

(b) Suppose that \( y[n] = h[n] \ast x[n] \), where

\[ x[n] = \begin{cases} 
1 & 0 \leq n \leq 9 \\
0 & \text{otherwise} 
\end{cases} \]

Use convolution to find \( y[n] \). You may find it useful to know that \( \sum_{n=0}^{L-1} a^n = \frac{1-a^L}{1-a} \).
3. (25 points) A linear shift-invariant system has the following frequency response:

\[ H(\omega) = e^{-5j\omega} \left( 1 + \frac{1}{2}\cos\omega \right) \]

(a) Suppose that \( y[n] = h[n] * x[n] \), where

\[ x[n] = 3\sin \left( \frac{\pi n}{6} \right) \]

What is \( y[n] \)?

(b) What is the impulse response, \( h[n] \), of this system?
4. (20 points) Suppose you want an FIR bandpass filter with a length of $N = 1024$, with cutoff frequencies $\omega_{LO} = 0.46\pi$ and $\omega_{HI} = 0.48\pi$, and with little stopband ripple. Find an impulse response $h[n]$ that meets these requirements.
5. (25 points) Consider the signal

\[ x[n] = \cos\left(\frac{\pi n}{100}\right) \]

(a) What is the 128-sample DFT, \( X[k] \), of this signal? Be sure to consider windowing effects.
(b) Suppose you want to construct the following periodic signal:

\[ y[n] = \cos\left( \frac{\pi(n - 128\ell)}{100} \right), \quad 128\ell \leq n < 128(\ell + 1), \text{ for all } \ell \]

You can create this signal using the following Fourier series:

\[ y[n] = \sum_{k=0}^{127} Y_k e^{j\frac{2\pi kn}{128}} \]

Notice that there is a relationship between \( Y_k \) and the DFT coefficients, \( X[k] \), that you computed in part (a) of this problem. Find \( Y_k \) in terms of \( X[k] \).
6. (25 points) Suppose you have a very long signal, \( x[n] \), that you want to filter to compute \( y[n] = h[n] \ast x[n] \). This can be done using the following sequence of steps:

\[
H[k] = \sum_{n=0}^{N-1} h[n]e^{-j\frac{2\pi kn}{N}}
\]

(1)

\[
x_\ell[n] = \begin{cases} 
  x[n + \ell M] & \text{if } 0 \leq n \leq M - 1, \text{ for all } \ell \\
  0 & \text{otherwise}
\end{cases}
\]

(2)

\[
X_\ell[k] = \sum_{n=0}^{N-1} x_\ell[n]e^{-j\frac{2\pi kn}{N}}
\]

(3)

\[
Y_\ell[k] = H[k]X_\ell[k]
\]

(4)

\[
y_\ell[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y_\ell[k]e^{j\frac{2\pi kn}{N}}
\]

(5)

\[
y[n] = \sum_{\ell=-\infty}^{\infty} y_\ell[n - \ell M]
\]

(6)

(a) Suppose \( h[n] \) is 129 samples long. Find values of \( M \) and \( N \) such that the algorithm in Eqs. (1) through (6) gives the same result as \( y[n] = h[n] \ast x[n] \). There are many different correct answers; you only need to find one correct answer.

(b) Suppose \( x[n] = \cos(0.08\pi n) \), and \( h[n] = \delta[n - 126] \). Find \( x_\ell[n] \) and \( y_\ell[n] \) in terms of \( \ell, M, \) and \( n \).
7. (20 points) Suppose that

\[ H(z) = \frac{1}{1 - 0.9e^{j\pi/6}z^{-1}} + \frac{1}{1 - 0.9e^{-j\pi/6}z^{-1}} \]

Find the pole(s) and zero(s) of \( H(z) \).
8. (20 points) Consider a notch filter with zeros at \( r_1 = e^{j0.47 \pi} \) and \( r_2 = e^{-j0.47 \pi} \), and with poles at \( p_1 = 0.999e^{j0.47 \pi} \) and \( p_2 = 0.999e^{-j0.47 \pi} \).

(a) What is the 3dB bandwidth of the notch, expressed in radians per sample?

(b) This filter can be implemented as

\[
y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]
\]

Find \( b_1, b_2, a_1, \) and \( a_2 \).
9. (20 points) A particular bell has a resonance at 440Hz, with a decay time of half a second, and another resonance at 1320Hz, with a decay time of 3 seconds. Find a filter, \( H(z) \), whose impulse response sounds like the impulse response of this bell if played through a D/A at \( F_s = 10,000 \) samples per second.