PRACTICE EXAM 3

Exam 3 will be held Tuesday, December 14, 8:00-11:00am

• This will a CLOSED BOOK exam.
• You will be permitted two sheets of handwritten notes, 8.5x11.
• Calculators and computers will not be permitted.
• Do not simplify explicit numerical expressions. The expression “\( e^{-5 \cos(3)} \)” is a MUCH better answer than “-0.00667”.
• If you’re taking the exam online, you will need to have your webcam turned on. Your exam will be sent to you by e-mail and on zoom at exactly 8:00am; you will need to photograph and upload your answers by exactly 11:00am.
• There will be a total of 200 points in the exam. Each problem specifies its point total. Plan your work accordingly.
• You must SHOW YOUR WORK to get full credit.

Name: __________________________
1. (20 points) Consider the following signal:

\[ x[n] = \delta[n - 15] + \delta[n - 30] \]

(a) \( X[k] \) is the 32-point DFT of \( x[n] \). Specify \( X[k] \) as a function of \( k \).

**Solution:**

\[ X[k] = e^{-j\frac{30\pi k}{32}} + e^{-j\frac{60\pi k}{32}} \]

(b) Suppose that \( h[n] \) is defined as follows:

\[ h[n] = \begin{cases} 
  e^{-n/14} & 0 \leq n \leq 14 \\
  0 & \text{otherwise} 
\end{cases} \]

Suppose that \( H[k] \) is the 32-point DFT of \( h[n] \), \( Y[k] = H[k]X[k] \), and \( y[n] \) is the inverse DFT of \( Y[k] \). Find \( y[n] \).

**Solution:**

\[ y[n] = \begin{cases} 
  e^{-(n+2)/14} & 0 \leq n \leq 12 \\
  e^{-(n-15)/14} & 15 \leq n \leq 29 \\
  e^{-(n-30)/14} & 30 \leq n \leq 31 
\end{cases} \]
2. (20 points) Consider the following system. The input of this system is $x[n]$, and the output is $y[n]$:

$v[n] = x[n] + 0.9v[n - 1]$
$y[n] = v[n] - 0.7y[n - 1]$

(a) What is the system function, $H(z)$, for this system?

Solution:

$$H(z) = \frac{1}{(1 - 0.9z^{-1})(1 + 0.7z^{-1})}$$

(b) What is the impulse response of this system?

Solution:

$$\frac{1}{(1 - 0.9z^{-1})(1 + 0.7z^{-1})} = \frac{C_1}{1 - 0.9z^{-1}} + \frac{C_2}{1 + 0.7z^{-1}}$$

$$1 = C_1(1 + 0.7z^{-1}) + C_2(1 - 0.9z^{-1})$$

$$1 = C_1(1 + 0.7/0.9) = C_1(16/9)$$

$$1 = C_2(1 - 0.9/(-0.7)) = C_2(16/7)$$

So $C_1 = \frac{9}{16}$, $C_2 = \frac{7}{16}$, and

$$h[n] = \left(\frac{9}{16}\right)(0.9)^n u[n] + \left(\frac{7}{16}\right)(-0.7)^n u[n]$$
3. (20 points) A flute player is playing a middle-A note. This system can be well modeled by blowing white noise through a damped resonator with a resonant frequency of 440Hz, and a bandwidth of 20Hz. Suppose you want to synthesize this flute digitally, by blowing white noise through a second-order damped resonator, with a sampling frequency of $F_s = 10,000\text{Hz}$.

(a) You want to implement the resonator as

$$y[n] = x[n] + a_1 y[n-1] + a_2 y[n-2]$$

What are $a_1$ and $a_2$?

**Solution:** The resonant frequency and half-bandwidth are

$$\omega_1 = \frac{2\pi 440}{10,000}$$
$$\sigma_1 = \frac{2\pi 10}{10,000}$$

So the system function is

$$H(z) = \frac{1}{\left(1 - e^{-\frac{2\pi 10}{10000}} + j \frac{2\pi 440}{10000} z^{-1}\right)\left(1 - e^{-\frac{2\pi 10}{10000}} - j \frac{2\pi 440}{10000} z^{-1}\right)}$$

$a_1 = (p_1 + p_1^*)$, and $a_2 = -|p_1|^2$, so

$$a_1 = 2e^{-\frac{2\pi 10}{10000}} \cos \left(\frac{2\pi 440}{10000}\right)$$
$$a_2 = -e^{-\frac{4\pi 10}{10000}}$$
(b) Suppose you have succeeded in implementing the digital filter. Now you give it the input $x[n] = \delta[n]$. What is the output?

**Solution:** The output is $h[n]$, which is

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1 (n + 1)) u[n]$$

where

$$\omega_1 = \frac{2\pi \cdot 440}{10,000}$$

$$\sigma_1 = \frac{2\pi \cdot 10}{10,000}$$
4. (20 points) You have recorded an electrocardiogram signal, $x[n]$, with a sampling frequency of $F_s = 1.2 kHz$. Unfortunately, it has been corrupted by power line noise: it has a big sinusoidal component at 60Hz. Fortunately, you know how to eliminate power line noise using a notch filter. All you have to do is to pass the signal through a difference equation:

$$y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$ (1)

Use a pole amplitude of 0.98. What are $b_1$, $b_2$, $a_1$, and $a_2$?

(Note: leave your answer in the form of an explicit numerical expression. For example, if you discover that $a_1 = (0.3)^2 \sin(2\pi/400)$, then you should leave it in that form instead of trying to simplify.)

Solution:

$$b_1 = -2 \cos(2\pi 60/1200)$$
$$b_2 = 1$$
$$a_1 = -2(0.98) \cos(2\pi 60/1200)$$
$$a_2 = (0.98)^2$$
5. (10 points) Consider the signal \( x(t) = -2 + \sin(40\pi t) \). Suppose we sample this signal at \( F_s = 100Hz \), then take a length-20 DFT of 20 samples of this signal. For which values of \( k \), \( 0 \leq k \leq 19 \), will the DFT samples \( X[k] \) be nonzero?

**Solution:** The discrete-time signal will be

\[
x[n] = -2 + \sin\left(\frac{40\pi n}{100}\right) = -2 + \sin(0.4\pi n)
\]

The DTFT is nonzero at frequencies \( \omega \in \{0, 0.4\pi, 1.6\pi\} \). The DTFT computes samples at \( \omega_k = \frac{2\pi k}{20} = 0.1\pi k \), so it is nonzero for \( k \in \{0, 4, 16\} \).
6. (20 points) Determine whether the following LTI system is causal and/or BIBO stable:

\[ y[n] = x[n + 1] + y[n - 1] \]

(a) Is it causal? Describe your reasoning in words.

**Solution:** No. \( y[n] \) depends on \( x[n + 1] \), which is in the future, therefore this system is not causal.

(b) Is it stable? Prove your answer.

**Solution:** No. The system function is

\[ H(z) = \frac{z}{1 - z^{-1}} \]

There is a pole at \( z = 1 \), which does not meet the criterion \( |z| < 1 \), therefore it is unstable.
7. (20 points) A particular system generates an output $y[n]$ from its input $x[n]$ according to the following rule:

$$y[n] = \begin{cases} 
x[n] & \text{n is even} \\
\frac{1}{2} (x[n - 1] + x[n + 1]) & \text{n is odd}
\end{cases}$$

(a) Is the system causal? Give your reason.

**Solution:** No. When $n$ is odd, $y[n]$ is the average of $x[n - 1]$ (which is in the past) and $x[n + 1]$ (which is in the future). Since it depends on a future input, it is not causal.

(b) Is the system stable? Give your reason.

**Solution:** Yes. For every finite input signal ($|x[n]| \leq M$ for some finite $M$), the corresponding output signal is also finite ($|y[n]| \leq M$ for the same $M$), so the system is stable.
8. (15 points) A signal is corrupted with narrowband noise at $\omega_n = \frac{\pi}{6}$ radians/sample. To remove the noise, you create a notch filter with a pole magnitude of $r = 0.9$. The filter system function is

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

Specify the poles and zeros in magnitude/phase form (e.g., $p_1 = me^{j\theta}$, but you should specify the numerical value of $m$ and the numerical value of $\theta$).

Solution:

- $r_1 = e^{j\pi/6}$
- $r_2 = e^{-j\pi/6}$
- $p_1 = 0.9e^{j\pi/6}$
- $p_2 = 0.9e^{-j\pi/6}$
9. (15 points) Suppose you want to implement a filter with the following frequency response:

\[ H(\omega) = \frac{(1 - 0.5e^{j\pi/4}e^{-j\omega})(1 - 0.5e^{-j\pi/4}e^{-j\omega})}{(1 + 0.5e^{j\pi/4}e^{-j\omega})(1 + 0.5e^{-j\pi/4}e^{-j\omega})} \]

You realize that you can get this frequency response by writing one line of code in MATLAB, implementing the following equation:

\[ y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2] \]

Find the filter coefficients.

<table>
<thead>
<tr>
<th>Solution:</th>
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<tbody>
<tr>
<td>( a_1 = \frac{-\sqrt{2}}{2} )</td>
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<tr>
<td>( a_2 = \frac{-1}{4} )</td>
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<tr>
<td>( b_1 = \frac{-\sqrt{2}}{2} )</td>
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<tr>
<td>( b_2 = \frac{1}{4} )</td>
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10. (30 points) Suppose you have a signal sampled at $F_s = 600$ samples/second. You wish to create a notch filter to eliminate a noise component at 60Hz. You choose to do this using the following filter:

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

(a) Specify the values of the poles $p_1, p_2$ and the zeros $r_1, r_2$. Note that there is one free parameter in your answer that is not specified by the problem statement; you may set that free parameter to any reasonable value.

**Solution:** $r_1 = e^{j\pi/5}$, $r_2 = e^{-j\pi/5}$, $p_1 = 0.99e^{j\pi/5}$, $p_2 = 0.99e^{-j\pi/5}$
(b) Now suppose you are given a system function

\[ H(z) = \frac{(1 - r_1 z^{-1})(1 - r_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} \]

and you wish to implement this using the equation

\[ y[n] = x[n] + b_1 x[n - 1] + b_2 x[n - 2] + a_1 y[n - 1] + a_2 y[n - 2] \]

Find \( b_1, b_2, a_1 \) and \( a_2 \) in terms of \( r_1, r_2, p_1 \) and \( p_2 \).

**Solution:** \( b_1 = -(r_1 + r_2), \ b_2 = r_1 r_2, \ a_1 = (p_1 + p_2), \ a_2 = -p_1 p_2 \)