Lecture 20: Discrete Fourier Transform

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Outline

Review: DTFT

The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n-n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X(\omega + 2\pi) = X(\omega)$
1. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

2. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
3. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
4. Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Outline

How can we compute the DTFT?

- The DTFT has a big problem: it requires an infinite-length summation, therefore you can't compute it on a computer.
- The DFT solves this problem by assuming a **finite length** signal.
- "N equations in N unknowns:" if there are N samples in the time domain ($x[n]$, $0 \le n \le N-1$), then there are only N independent samples in the frequency domain ($X(\omega_k)$, $0 \le k \le N-1$).

Finite-length signal

First, assume that $x[n]$ is nonzero only for $0 \leq n \leq N-1$. Then the DTFT can be computed as:

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

N equations in N unknowns

Since there are only N samples in the time domain, there are also only N **independent** samples in the frequency domain:

$$X[k] = X(\omega_k) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

where

$$\omega_k = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1$$

Discrete Fourier Transform

Putting it all together, we get the formula for the DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

Inverse Discrete Fourier Transform

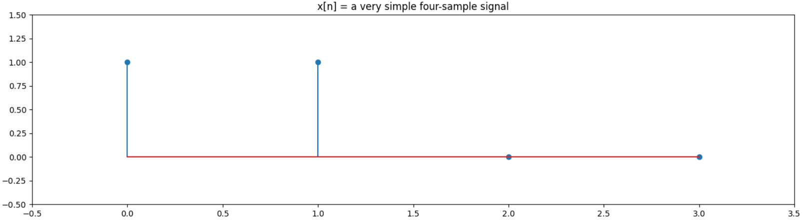
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

Using orthogonality, we can also show that

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

Outline

Example



Consider the signal

$$x[n] = \begin{cases} 1 & \text{n=0,1} \\ 0 & \text{n=2,3} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Example DFT

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi kn}{4}} \\
&= 1 + e^{-j\frac{2\pi k}{4}} \\
&= \begin{cases} 2 & k = 0 \\ 1 - j & k = 1 \\ 0 & k = 2 \\ 1 + j & k = 3 \end{cases}
\end{aligned}
$$

Example IDFT

$$X[k] = [2, (1-j), 0, (1+j)]$$

$$\begin{aligned} x[n] &= \frac{1}{4}\sum_{k=0}^{3} X[k] e^{j\frac{2\pi kn}{4}} \\ &= \frac{1}{4}\left(2 + (1-j)e^{j\frac{2\pi n}{4}} + (1+j)e^{j\frac{6\pi n}{4}}\right) \\ &= \frac{1}{4}\left(2 + (1-j)j^n + (1+j)(-j)^n\right) \\ &= \begin{cases} 1 & n = 0,1 \\ 0 & n = 2,3 \end{cases} \end{aligned}$$

Outline

Shifted Delta Function

In many cases, we can find the DFT directly from the DTFT. For example:

$$h[n]=\delta[n-n_0] \quad \leftrightarrow \quad H(\omega)=e^{-j\omega n_0}$$

If and only if the signal is less than length N, we can just plug in $\omega_k=\frac{2\pi k}{N}$:

$$h[n]=\delta[n-n_0] \quad \leftrightarrow \quad H[k]=\begin{cases} e^{-j\frac{2\pi k n_0}{N}} & 0\le n_0\le N-1 \\ \text{undefined} & \text{otherwise}\end{cases}$$

Outline

Cosine

Finding the DFT of a cosine is possible, but harder than you might think. Consider:

$$x[n] = \cos(\omega_0 n)$$

This signal violates the first requirement of a DFT:

- $x[n]$ must be finite length.

Cosine

We can make $x[n]$ finite-length by windowing it, like this:

$$x[n] = \cos(\omega_0 n) w[n],$$

where $w[n]$ is the rectangular window,

$$w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Cosine

Now that $x[n]$ is finite length, we can just take its DTFT, and then sample at $\omega_k = \frac{2\pi k}{N}$:

$$X[k] = X(\omega_k) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n}$$

Linearity and Frequency-Shift Properties of the DTFT

But how do we solve this equation?

$$X(\omega_k) = \sum_{n=0}^{N-1} \cos(\omega_0 n) w[n] e^{-j\omega_k n}$$

The answer is, surprisingly, that we can use two properties of the DTFT:

- **Linearity:** $x_1[n] + x_2[n] \quad \leftrightarrow \quad X_1(\omega) + X_2(\omega)$
- **Frequency Shift:** $e^{j\omega_0 n} z[n] \quad \leftrightarrow \quad Z(\omega - \omega_0)$

Linearity and Frequency-Shift Properties of the DTFT

- **Linearity:**

$$\cos(\omega_0 n)w[n] = \frac{1}{2}e^{j\omega_0 n}w[n] + \frac{1}{2}e^{-j\omega_0 n}w[n]$$

- **Frequency Shift:**

$$e^{j\omega_0 n}w[n] \leftrightarrow W(\omega - \omega_0)$$

Putting them together, we have that

$$\cos(\omega_0 n)w[n] \quad \leftrightarrow \quad \frac{1}{2}W(\omega - \omega_0) + \frac{1}{2}W(\omega + \omega_0)$$

DFT of a Cosine

Putting it together,

$$x[n] = \cos(\omega_0 n) w[n] \quad \leftrightarrow \quad X(\omega_k) = \frac{1}{2} W(\omega_k - \omega_0) + \frac{1}{2} W(\omega_k + \omega_0)$$

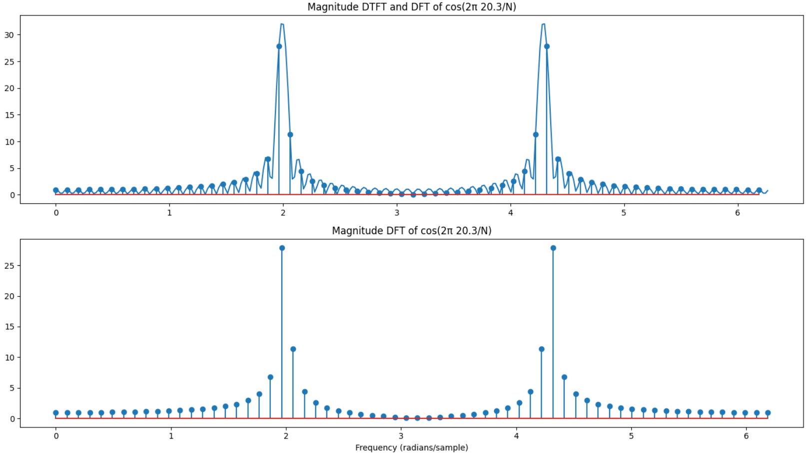
where $W(\omega)$ is the Dirichlet form:

$$W(omega) = e^{-j\omega \frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

DFT of a Cosine

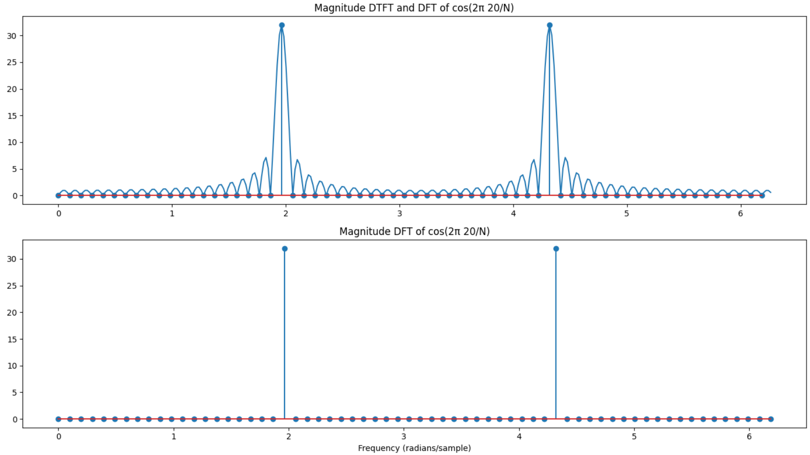
Here's the DFT of

$$x[n] = \cos\left(\frac{2\pi 20.3}{N}\right) w[n]$$



DFT of a Cosine

Remember that $W(\omega) = 0$ whenever ω is a multiple of $\frac{2\pi}{N}$. But **the DFT only samples at multiples of** $\frac{2\pi}{N}$! So if ω_0 is **also** a multiple of $\frac{2\pi}{N}$, then the DFT of a cosine is just a pair of impulses in frequency:



Outline

Periodic in Frequency

Just as $X(\omega)$ is periodic with period 2π, in the same way, $X[k]$ is periodic with period N:

$$\begin{aligned} X[k+N] &= \sum_n x[n] e^{-j\frac{2\pi(k+N)n}{N}} \\ &= \sum_n x[n] e^{-j\frac{2\pi kn}{N}} e^{-j\frac{2\pi Nn}{N}} \\ &= \sum_n x[n] e^{-j\frac{2\pi kn}{N}} \\ &= X[k] \end{aligned}$$

Periodic in Time

The inverse DFT is also periodic in time! $x[n]$ is undefined outside $0 \leq n \leq N-1$, but if we accidentally try to compute $x[n]$ at any other times, we end up with:

$$\begin{aligned} x[n+N] &= \frac{1}{N}\sum_k X[k]e^{j\frac{2\pi k(n+N)}{N}} \\ &= \frac{1}{N}\sum_k X[k]e^{j\frac{2\pi kn}{N}}e^{j\frac{2\pi kN}{N}} \\ &= \frac{1}{N}\sum_k X[k]e^{j\frac{2\pi kn}{N}} \\ &= x[n] \end{aligned}$$

Linearity

$$ax_1[n] + bx_2[n] \quad \leftrightarrow \quad aX_1[k] + bX_2[k]$$

Samples of the DTFT

If $x[n]$ is finite length, with length of at most N samples, then

$$X[k] = X(\omega_k), \quad \omega_k = \frac{2\pi k}{N}$$

Conjugate Symmetry of the DTFT

Here's a property of the DTFT that we didn't talk about much. Suppose that $x[n]$ is real. Then

$$
\begin{aligned}
X(-\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(-\omega)n} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} \\
&= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)^* \\
&= X^*(\omega)
\end{aligned}
$$

Conjugate Symmetry of the DFT

$$X(\omega) = X^*(-\omega)$$

Remember that the DFT, $X[k]$, is just the samples of the DTFT, sampled at $\omega_k = \frac{2\pi k}{N}$. So that means that conjugate symmetry also applies to the DFT:

$$X[k] = X^*[-k]$$

But remember that the DFT is periodic with a period of N, so

$$X[k] = X^*[-k] = X^*[N-k]$$

Frequency Shift

The frequency shift property of the DTFT also applies to the DFT:

$$w[n]e^{j\omega_0 n} \quad \leftrightarrow \quad W(\omega - \omega_0)$$

If $\omega = \frac{2\pi k}{N}$, and if $\omega_0 = \frac{2\pi k_0}{N}$, then we get

$$w[n]e^{j\frac{2\pi k_0 n}{N}} \quad \leftrightarrow \quad W[k - k_0]$$

Time Shift

The time shift property of the DTFT was

$$x[n-n_0] \quad \leftrightarrow \quad e^{j\omega n_0}X(\omega)$$

The same thing also applies to the DFT, except that **the DFT is finite in time**. Therefore we have to use what's called a "circular shift:"

$$x\left[((n-n_0))_N\right] \quad \leftrightarrow \quad e^{j\frac{2\pi k n_0}{N}}X[k]$$

where $((n-n_0))_N$ means "$n-n_0$, modulo N." We'll talk more about what that means in the next lecture.

Outline

DFT Examples

1. $$x[n] = [1,1,0,0] \quad \leftrightarrow \quad X[k] = [2, 1-j, 0, 1+j]$$

2. $$x[n] = \delta[n-n_0] \quad \leftrightarrow \quad X[k] = \begin{cases} e^{-j\frac{2\pi k n_0}{N}} & 0 \le n_0 \le N-1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

3. $$x[n] = w[n]\cos(\omega_0 n)$$
$$\leftrightarrow \quad X[k] = \frac{1}{2} W\left[k - \frac{N\omega_0}{2\pi}\right] + \frac{1}{2} W\left[k + \frac{N\omega_0}{2\pi}\right]$$

DFT Properties

1. **Periodic in Time and Frequency:**

$$x[n] = x[n+N], \qquad X[k] = X[k+N]$$

2. **Linearity:**

$$ax_1[n] + bx_2[n] \quad \leftrightarrow \quad aX_1[k] + bX_2[k]$$

3. **Samples of the DTFT:** if $x[n]$ has length at most N samples, then

$$X[k] = X(\omega_k), \quad \omega_k = \frac{2\pi k}{N}$$

4. **Frequency Shift:**

$$x[n] = e^{j\frac{2\pi k_0 n}{N}} \quad \leftrightarrow \quad X[k-k_0]$$

Outline

Written Example

Show that the signal $x[n] = \delta[n - n_0]$ obeys the conjugate symmetry properties of both the DFT and DTFT.