Lecture 10: Ideal Filters

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

Review: Ideal Filters

- Ideal Lowpass Filter:

$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| \le \omega_c, \\ 0 & \omega_c < |\omega| \le \pi. \end{cases} \quad \leftrightarrow \quad h_{LP}[m] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Highpass Filter:

$$H_{HP}(\omega) = 1 - H_{LP}(\omega) \quad \leftrightarrow \quad h_{HP}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$$

- Ideal Bandpass Filter:

$$H_{BP}(\omega) = H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega)$$
$$\leftrightarrow h_{BP}[n] = \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n)$$

Outline

Ideal Filters are Infinitely Long

- All of the ideal filters, $h_{LP,i}[n]$ and so on, are infinitely long!
- In demos so far, I've faked infinite length by just making $h_{LP,i}[n]$ more than twice as long as $x[n]$.
- If $x[n]$ is very long (say, a 24-hour audio recording), you probably don't want to do that (computation=expensive)

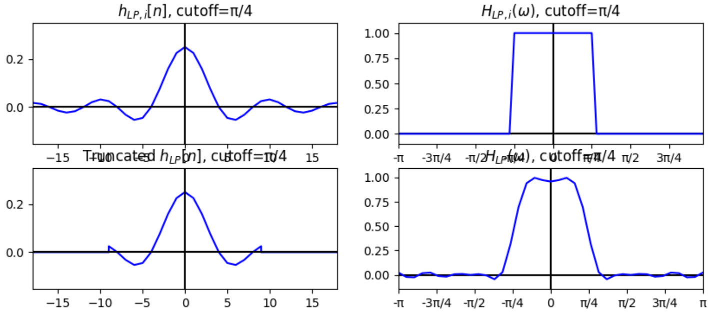
Finite Length by Truncation

We can force $h_{LP,i}[n]$ to be finite length by just truncating it, say, to $2M+1$ samples:

$$h_{LP}[n] = \begin{cases} h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

Truncation Causes Frequency Artifacts

The problem with truncation is that it causes artifacts.



Windowing Reduces the Artifacts

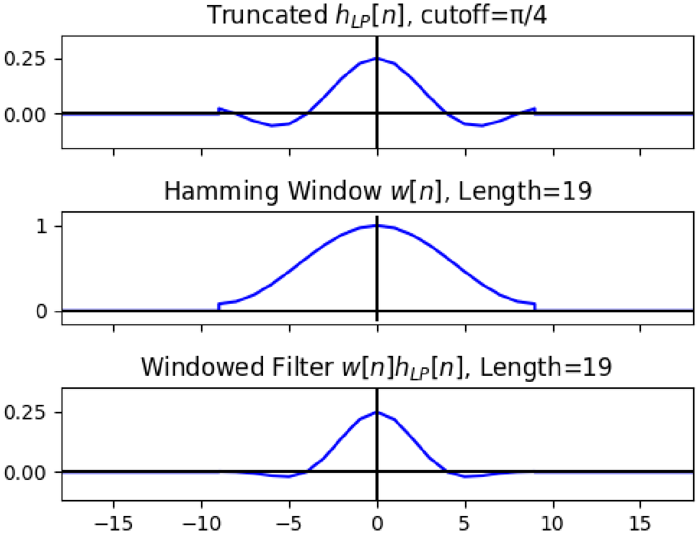
We can reduce the artifacts (a lot) by windowing $h_{LP,i}[n]$, instead of just truncating it:

$$h_{LP}[n] = \begin{cases} w[n]h_{LP,i}[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

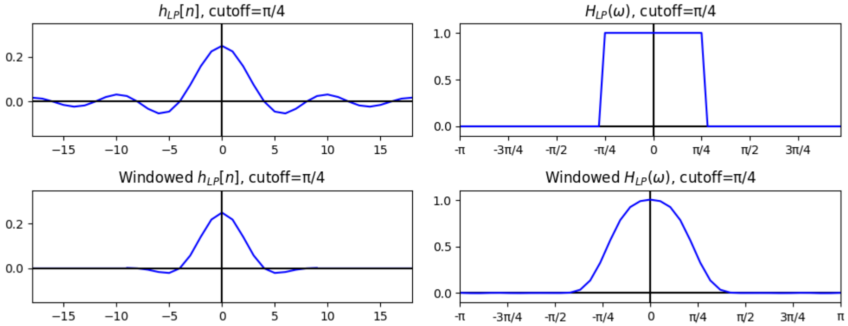
where $w[n]$ is a window that tapers smoothly down to near zero at $n = \pm M$, e.g., a Hamming window:

$$w[n] = 0.54 + 0.46 \cos\left(\frac{2\pi n}{2M}\right)$$

Windowing a Lowpass Filter



Windowing Reduces the Artifacts



Outline

Even Length Filters

Often, we'd like our filter $h_{LP}[n]$ to be even length, e.g., 200 samples long, or 256 samples. We can't do that with this definition:

$$h_{LP}[n] = \begin{cases} w[n]h_{LP,i}[n] & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

…because $2M+1$ is always an odd number.

Even Length Filters using Delay

We can solve this problem using the time-shift property of the DTFT:

$$z[n] = x[n - n_0] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega n_0} X(\omega)$$

Even Length Filters using Delay

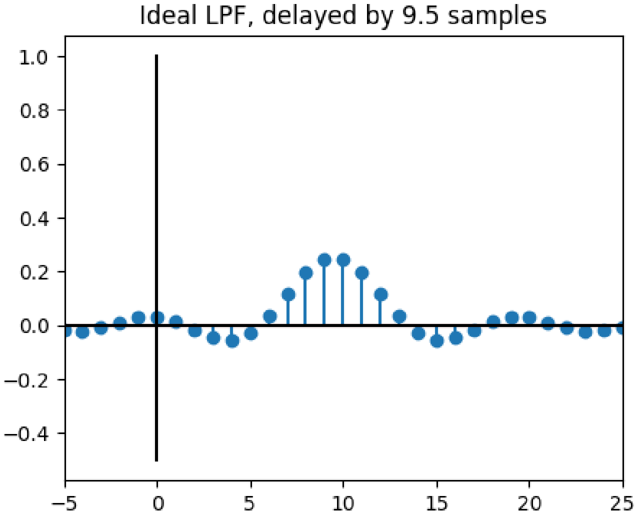
Let's delay the ideal filter by exactly $M-0.5$ samples, for any integer M:

$$z[n]=h_{LP,i}\left[n-(M-0.5)\right]=\frac{\omega_c}{\pi}\text{sinc}\left(\omega\left(n-M+\frac{1}{2}\right)\right)$$

I know that sounds weird. But notice the symmetry it gives us. The whole signal is symmetric w.r.t. sample $n=M-0.5$. So $z[M-1]=z[M]$, and $z[M-2]=z[M+1]$, and so one, all the way out to

$$z[0]=z[2M-1]=\frac{\omega_c}{\pi}\text{sinc}\left(\omega\left(M-\frac{1}{2}\right)\right)$$

Even Length Filters using Delay



Even Length Filters using Delay

Apply the time delay property:

$$z[n] = h_{LP,i}\left[n - (M - 0.5)\right] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega(M-0.5)} H_{LP,i}(\omega),$$

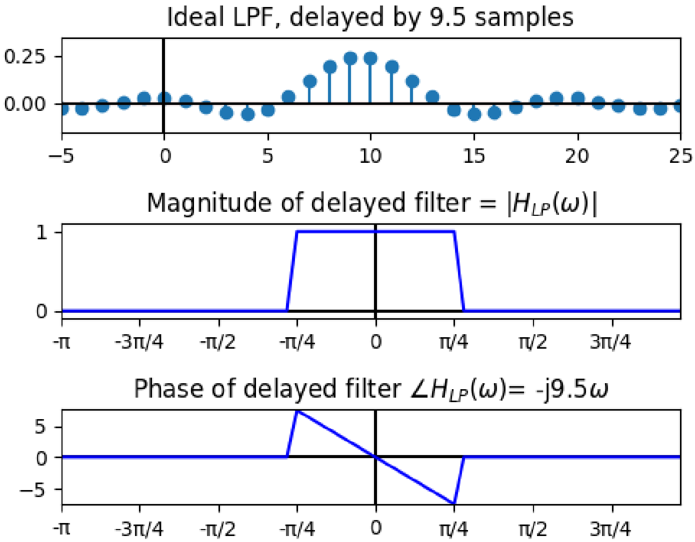
and then notice that

$$\left|e^{-j\omega(M-0.5)}\right| = 1$$

So

$$|Z(\omega)| = |H_{LP,i}(\omega)|$$

Even Length Filters using Delay



Even Length Filters using Delay and Windowing

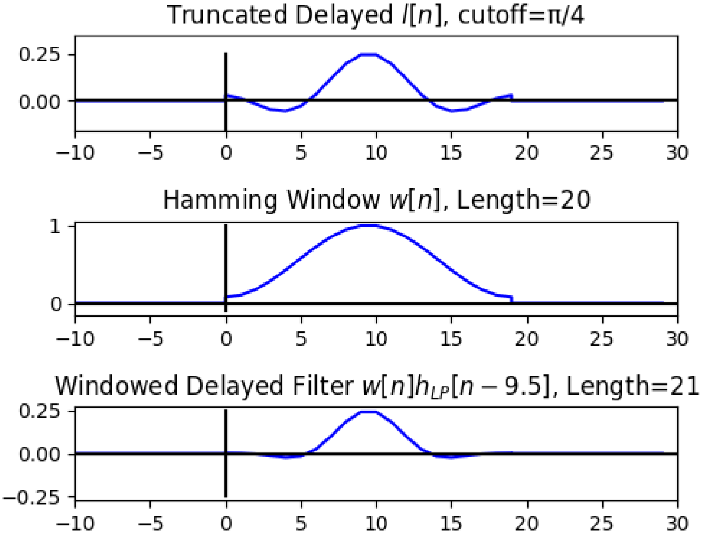
Now we can create an even-length filter by windowing the delayed filter:

$$h_{LP}[n] = \begin{cases} w[n] h_{LP,i}\left[n-(M-0.5)\right] & 0 \le n \le (2M-1) \\ 0 & \text{otherwise} \end{cases}$$

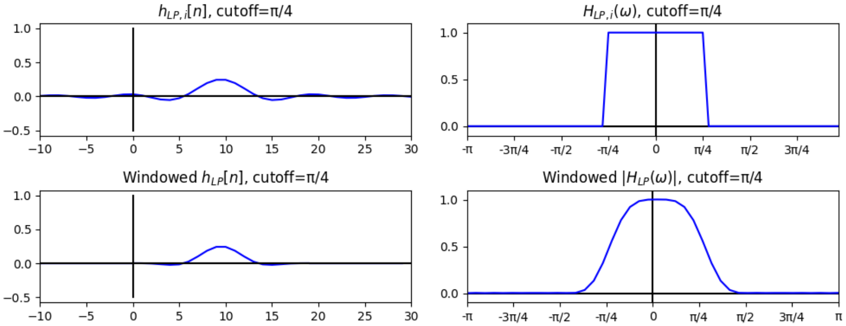
where $w[n]$ is a Hamming window defined for the samples $0 \le m \le 2M-1$:

$$w[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{2M-1}\right)$$

Even Length Filters using Delay and Windowing



Even Length Filters using Delay and Windowing



Outline

Summary: Ideal Filters

- Ideal Lowpass Filter:

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- Ideal Highpass Filter:

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- Ideal Bandpass Filter:

$$\begin{aligned} H_{BP}(\omega) &= H_{LP,\omega_2}(\omega) - H_{LP,\omega_1}(\omega) \\ \leftrightarrow h_{BP}[n] &= \frac{\omega_2}{\pi}\text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi}\text{sinc}(\omega_1 n) \end{aligned}$$

Summary: Practical Filters

- Odd Length:

$$h_{HP}[n] = \begin{cases} h_{HP,i}[n]w[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

- Even Length:

$$h_{HP}[n] = \begin{cases} h_{HP,i}\left[n-(M-0.5)\right]w[n] & 0 \leq n \leq 2M-1 \\ 0 & \text{otherwise} \end{cases}$$

where $w[n]$ is a window with tapered ends, e.g.,

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{L-1}\right) & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

Outline

Written Example

Design a bandpass filter with lower and upper cutoffs of $\omega_1 = \frac{\pi}{3}$, $\omega_2 = \frac{\pi}{2}$, and with a length of $N = 33$ samples, using a Hamming window.