Lecture 10: Exam 1 Sample Problems

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis, Fall 2021

Outline

Question

Calculate the Fourier series coefficients X_0 and X_k for the periodic signal $x(t) = x(t+8)$:

$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ -1, & 1 \leq t \leq 3 \\ 0, & 3 < t < 8 \end{cases}$$

Answer Part 1

Calculate the Fourier series coefficients X_0 and X_k for the periodic signal $x(t) = x(t+8)$:

$$
\begin{aligned}
X_0 &= \frac{1}{8}\int_0^8 x(t)dt \\
&= \frac{1}{8}\left(\int_0^1 dt - \int_1^3 dt\right) \\
&= -\frac{1}{8}
\end{aligned}
$$

Answer Part 2

Calculate the Fourier series coefficients X_0 and X_k for the periodic signal $x(t) = x(t+8)$:

$$\begin{aligned} X_k &= \frac{1}{8}\int_0^8 x(t)e^{-j2\pi kt/8}dt \\ &= \frac{1}{8}\left(\int_0^1 e^{-j2\pi kt/8}dt - \int_1^3 e^{-j2\pi kt/8}dt\right) \\ &= \frac{1}{8}\left(\frac{1}{-j2\pi k/8}\right)\left(\left[e^{-j2\pi kt/8}\right]_0^1 - \left[e^{-j2\pi kt/8}\right]_1^3\right) \\ &= \left(\frac{1}{-j2\pi k}\right)\left(2e^{-j2\pi k/8} - 1 - e^{-j6\pi k/8}\right) \end{aligned}$$

Outline

Question

Suppose that we have a signal bandlimited to 5kHz. What is the minimum F_s necessary to avoid aliasing?

Answer

10kHz

Outline

Question

Assume that $x[n] = x_c(nT)$, where $1/T = 10,000$ samples/second. Find $x[n]$ and its spectrum if

$$x_c(t) = \cos(7000\pi t)$$

Answer

$$x[n] = \cos\left(\frac{7000\pi n}{10,000}\right)$$

...and the spectrum is

$$\left\{(-\frac{7000\pi}{10000}, \frac{1}{2}), (\frac{7000\pi}{10000}, \frac{1}{2})\right\}$$

Outline

Part (a)

Consider the signal $x(t) = -2 + \sin(40\pi t)$. Determine and list all of the analog frequencies in the signal $x(t)$. Include negative frequencies.

Answer

{-20,0,20}

Part (b)

$x(t) = -2 + \sin(40\pi t)$. What is the lowest possible sampling frequency that would avoid aliasing?

Answer

$F_s > 2f = 40$

Part (c)

What is the corresponding Nyquist frequency for the sampling rate you found in part (b)?

Answer

$F_N = \frac{F_s}{2} > 20$

Part (d)

$x(t) = -2 + \sin(40\pi t)$. For a sampling frequency of $F_s = 100$Hz, find $x[n]$.

Answer

$$x[n] = -2 + \sin\left(\frac{40\pi n}{100}\right)$$

Part (e)

$x(t) = -2 + \sin(40\pi t)$, $F_s = 100$Hz. Determine and list all of the frequencies ω, $-\pi < \omega \leq \pi$, present in the discrete-time signal $x[n]$. Include negative frequencies.

Answer

$$\left\{ -\frac{40\pi}{100}, 0, \frac{40\pi}{100} \right\}$$

Outline

Question

$$\cos(\omega t) + \cos(\omega t + \frac{\pi}{3}) = m\cos(\omega t + \theta)$$

Find x and y such that $m = \sqrt{x^2 + y^2}$ and $\theta = \text{atan2}(x, y)$, the two-argument arctangent of x and y.

Answer

$$\begin{aligned}\cos(\omega t) + \cos(\omega t + \frac{\pi}{3}) &= \Re\left\{(1 + e^{j\pi/3})e^{j\omega t}\right\} \\ &= \Re\left\{(1 + \cos(\pi/3) + j\sin(\pi/3))e^{j\omega t}\right\}\end{aligned}$$

So

$$x = 1 + \cos(\pi/3), \quad y = \sin(\pi/3)$$

Outline

Question

A signal $x(t) = \cos(2\pi 6000t)$ is sampled at $F_s = 8000$ samples/second to create $y[n]$. The digital signal $y[n]$ is then played back through an ideal D/A at the same sampling rate, $F_s = 8000$ samples/second, to generate a signal $z(t)$. Find $z(t)$.

Answer

$$x(t) = \cos(2\pi 6000 t)$$

$$y[n] = \cos\left(\frac{2\pi 6000 n}{8000}\right)$$

$$= \cos\left(\frac{3\pi n}{2}\right)$$

$$= \cos\left(\frac{\pi n}{2}\right)$$

$$z(t) = \cos\left(\frac{\pi}{2} 8000 t\right) = \cos(4000\pi t)$$

Outline

Question

The signal $x[n]$ is periodic with period $N_0 = 4$. Its values in each period are

$$x[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1, 2, 3 \end{cases}$$

Find the Fourier series coefficients.

Answer

$$
\begin{aligned}
X_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi kn/4} \\
&= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi kn/2} \\
&= \frac{1}{4}\left(1 - e^{-j\pi k/2} - e^{-j\pi k} - e^{-j\pi 3k/2}\right)
\end{aligned}
$$

Outline

Question

$$6\cos\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)+6\sin\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)$$
$$=A\cos(\Omega t+\phi)$$

Find A, Ω, and ϕ.

Answer

$$
\begin{aligned}
&6\cos\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)+6\sin\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)\\
&=6\cos\left(2\pi 1000\left(t-\frac{1}{4000}\right)\right)+6\cos\left(2\pi 1000\left(t-\frac{1}{4000}\right)-\frac{\pi}{2}\right)\\
&=6\cos\left(2\pi 1000t-\frac{\pi}{2}\right)+6\cos\left(2\pi 1000t-\frac{\pi}{2}-\frac{\pi}{2}\right)\\
&=\Re\left\{6(e^{-j\pi/2}+e^{-j\pi})e^{j2000\pi t}\right\}\\
&=\Re\left\{6(-j-1)e^{j2000\pi t}\right\}\\
&=\Re\left\{6\sqrt{2}e^{-j3\pi/4}e^{j2000\pi t}\right\}
\end{aligned}
$$

So $A=6\sqrt{2}$, $\Omega=2000\pi$, $\phi=-\frac{3\pi}{4}$.

Outline

Question

A periodic signal $x(t)$, with period T_0, is given by

$$x(t) = \begin{cases} 1 & 0 \leq t \leq \frac{3T_0}{4} \\ 0 & \frac{3T_0}{4} < t < T_0 \end{cases}$$

The same signal can be expressed as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Find $|X_2|$, the amplitude of the second harmonic.

Answer

$$
\begin{aligned}
X_2 &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi 2t/T_0}dt \\
&= \frac{1}{T_0}\int_0^{3T_0/4} e^{-j4\pi t/T_0}dt \\
&= \frac{1}{T_0}\left(\frac{1}{-j4\pi/T_0}\right)\left[e^{-j4\pi t/T_0}\right]_0^{3T_0/4} \\
&= \left(\frac{1}{-j4\pi}\right)\left(e^{-j3\pi}-1\right) = \left(\frac{-2}{-j4\pi}\right)
\end{aligned}
$$

So $|X_2| = 1/2\pi$.

Outline

Question

An 8000Hz tone, $x(t) = \cos(2\pi 8000t)$, is sampled at $F_s = \frac{1}{T} = 10,000$ samples/second in order to create $x[n] = x(nT)$. Sketch $X(\omega)$ for $0 \leq \omega \leq 2\pi$ (**note the domain!!**). Specify the frequencies at which $X(\omega) \neq 0$.

Answer

Answer should be a spectrum plot with spikes at $\omega = 8\pi/5$ and $\omega = 2\pi/5$, each labeled with a phasor of $1/2$.

Outline

Question, Part 1

Each of the following is sampled at $F_s = 10000$ samples/second, producing either $x[n] =$constant, or $x[n] = \cos\omega n$ for some value of ω. Specify the constant if possible; otherwise, specify ω such that $-\pi \le \omega < \pi$.

$$x(t) = \cos\left(2\pi 900 t\right)$$

Answer

Solution: $\omega = \frac{1800\pi}{10{,}000}$

Question, Part 2

Each of the following is sampled at $F_s = 10000$ samples/second, producing either $x[n]$ =constant, or $x[n] = \cos \omega n$ for some value of ω. Specify the constant if possible; otherwise, specify ω such that $-\pi \leq \omega < \pi$.

$$x(t) = \cos\left(2\pi 10000 t\right)$$

Answer

Solution: $x[n] = 1$

Question, Part 3

Each of the following is sampled at $F_s = 10000$ samples/second, producing either $x[n]$ =constant, or $x[n] = \cos \omega n$ for some value of ω. Specify the constant if possible; otherwise, specify ω such that $-\pi \leq \omega < \pi$.

$$x(t) = \cos(2\pi 11000 t)$$

Answer

Solution: $\omega = \frac{22000\pi}{10000} - 2\pi = \frac{2000\pi}{10000}$

Outline

Question

Consider the signal

$$x(t) = 2\cos(2\pi 440t) - 3\sin(2\pi 440t)$$

This signal can also be written as $x(t) = A\cos(\omega t + \theta)$ for some $A = \sqrt{M}$, ω, and $\theta = \text{atan}(R)$. Find M, ω, and R.

Answer

$$\begin{aligned} x(t) &= 2\cos\left(2\pi 440 t\right) - 3\sin\left(2\pi 440 t\right) \\ &= 2\cos\left(2\pi 440 t\right) - 3\cos\left(2\pi 440 t - \frac{\pi}{2}\right) \\ &= \Re\left\{(2 - 3e^{-j\pi/2})e^{j2\pi 440 t}\right\} \\ &= \Re\left\{(2 + 3j)e^{j2\pi 440 t}\right\} \\ &= \Re\left\{\sqrt{5}e^{j\mathrm{atan}(3/2)}e^{j2\pi 440 t}\right\} \end{aligned}$$

So $A = \sqrt{13}$, $\omega = 2\pi 440$, and $\theta = \mathrm{atan}(3/2)$.

Outline

Question, Part 1

A signal $x(t)$ is periodic with $T_0 = 0.02$ seconds, and its values are specified by

$$x(t) = \begin{cases} -1 & 0 \leq t \leq 0.01 \\ 0 & 0.01 < t < 0.02 \end{cases}$$

Sketch $x(t)$ as a function of t for $0 \leq t \leq 0.02$ seconds. Label at least one important tic mark, each, on the horizontal and vertical axes.

Answer

Sketch should show $x(t) = -1$ between 0 and 0.01, then $x(t) = 0$ between 0.01 and 0.02.

Question, Part 2

A signal $x(t)$ is periodic with $T_0 = 0.02$ seconds, and its values are specified by

$$x(t) = \begin{cases} -1 & 0 \leq t \leq 0.01 \\ 0 & 0.01 < t < 0.02 \end{cases}$$

What is F_0?

Answer

$$F_0 = \frac{1}{T_0} = \frac{1}{0.02}$$

Question, Part 3

A signal $x(t)$ is periodic with $T_0 = 0.02$ seconds, and its values are specified by

$$x(t) = \begin{cases} -1 & 0 \le t \le 0.01 \\ 0 & 0.01 < t < 0.02 \end{cases}$$

Find X_0 without doing any integral.

Answer

$x(t)$ is -1 for half a period, and 0 for half a period, so its average value is $X_0 = -\frac{1}{2}$.

Question, Part 3

A signal $x(t)$ is periodic with $T_0 = 0.02$ seconds, and its values are specified by

$$x(t) = \begin{cases} -1 & 0 \leq t \leq 0.01 \\ 0 & 0.01 < t < 0.02 \end{cases}$$

Find X_k for all the other values of k, i.e., for $k \neq 0$. Simplify; your answer should have no exponentials in it.

Answer

$$\begin{aligned}
X_k &= \frac{1}{0.02}\int_0^{0.02} x(t)e^{-j2\pi kt/0.02}dt \\
&= \frac{1}{0.02}\int_0^{0.01} e^{-j2\pi kt/0.02}dt \\
&= \frac{1}{0.02}\left(\frac{1}{-j2\pi k/0.02}\right)\left[e^{-j2\pi kt/0.02}\right]_0^{0.01} \\
&= \left(\frac{1}{-j2\pi k}\right)\left(e^{-j2\pi k 0.01/0.02} - 1\right) \\
&= \left(\frac{1}{-j2\pi k}\right)\left(e^{-j\pi k} - 1\right) \\
&= \left(\frac{1}{-j2\pi k}\right)\left((-1)^k - 1\right)
\end{aligned}$$

Outline

Question

In order to become a billionaire, you've decided you need to know what was the total value of the U.S. GDP every day of every year since 1901. Unfortunately, GDP figures are only published once per year (once per 365 days), so you need to interpolate them.

Consider the following system:

$$d[n] = \sum_{m=-\infty}^{\infty} y[m]g[n-365m] \tag{1}$$

where $y[m]$ is the GDP in the m^{th} year, and $d[n]$ is the estimated GDP in the n^{th} day.

Design the filter $g[n]$ so that Eq. 1 implements **PIECE-WISE LINEAR** interpolation. (Draw a sketch of $g[n]$ that specifies the values of all of its samples, or write a formula that does so).

Answer

$$g[n] = \begin{cases} 1 - \frac{|n|}{365} & -365 \leq n \leq 365 \\ 0 & \text{otherwise} \end{cases}$$