

ECE 401 Signal and Image Analysis

Homework 3

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: Monday, 9/27/2021; Due: Tuesday, 10/5/2021
Reading: *DSP First* Chapter 5

Problem 3.1

In MP3, one of the filters you'll create is a local averaging filter. A local averaging filter produces an output $y[n]$, at time n , which is the average of the previous N samples of $x[m]$:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \quad (3.1-1)$$

$$h[m] = \begin{cases} \frac{1}{N} & 0 \leq m \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad (3.1-2)$$

- (a) First, consider what happens if $x[m]$ is a pure tone with a period of $N_0 = \frac{2\pi}{\omega_0}$, an amplitude of A , and a phase of θ :

$$x[n] = A \cos(\omega_0 n - \theta)$$

Suppose that the averaging window, N , is exactly an integer multiple of N_0 . For example, suppose that $N = 3N_0$. Draw a picture of $x[n]$ as a function of n , and shade in the regions that would be added together by the summation in Eq. (3.1-1) in order to compute $y[0]$. Argue based on your figure (with no calculations at all) that $y[0] = 0$.

Solution: The picture should show that we are adding together three complete periods of the cosine up through, and including, the sample $x[0]$. Every period of the cosine has a positive section and a negative section. When we average these two sections, they cancel each other out.

- (b) Adding up the samples of a cosine is easy when N is an integer multiple of N_0 , but hard otherwise. It's actually much easier to add the samples of a complex exponential, because we can use the standard geometric series formula (https://en.wikipedia.org/wiki/Geometric_series#Formula). Use that formula to find $y[N-1]$ when

$$x[n] = e^{j\omega_0 n}$$

Your result should have the form $y[0] = (1-a)/(1-b)$ for some complex-valued constants, a and b , that depend on π , N , and ω_0 , but not on m or n .

Solution:

$$y[0] = \frac{1 - e^{-j\omega_0 N}}{1 - e^{-j\omega_0}}$$

Problem 3.2

Another of the filters you'll create in MP3 is a backward-difference filter. A backward-difference filter is one of several different ways of approximating a first-derivative:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \quad (3.2-1)$$

$$h[m] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.2-2)$$

- (a) First, consider what happens if $x[m]$ is a pure tone with a period of $T_0 = \frac{2\pi}{\omega_0}$, an amplitude of A , and a phase of θ :

$$x[n] = A \cos(\omega_0 n - \theta) \quad (3.2-3)$$

Plug Eq. (3.2-3) into Eq. (3.2-1), then use the following trigonometric identity and the following approximation to discover in exactly what way the first-difference operator approximates a derivative:

$$\begin{aligned} -2 \sin(a) \sin(b) &= \cos(a+b) - \cos(a-b) \\ \sin(b) &\approx b \text{ if } b \text{ is small} \end{aligned}$$

In order to apply the approximation, you can assume that ω_0 is a small number.

Solution: Plugging Eq. (3.2-3) into Eq. (3.2-1) gives

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= A \cos(\omega_0 n - \theta) - A \cos(\omega_0(n-1) - \theta) \end{aligned}$$

We can apply the trig identity if we set

$$\begin{aligned} a &= \omega_0(n-0.5) - \theta \\ b &= 0.5\omega_0 \end{aligned}$$

which gives

$$y[n] = -2A \sin(0.5\omega_0) \sin(\omega_0(n-0.5) + \theta)$$

Applying the approximation gives

$$y[n] \approx -A\omega_0 \sin(\omega_0(n-0.5) + \theta),$$

which is what you would get if you differentiated $A \cos(\omega_0(n-0.5) + \theta)$ with respect to n .

- (b) Let's try the same thing with a complex exponential. Plug the following value of $x[n]$ into Eq. (3.2-1)

$$x[n] = e^{j\omega_0 n},$$

then assume that ω_0 is a small number, and simplify using the approximation

$$e^\phi \approx 1 + \phi \text{ if } \phi \text{ is small}$$

in order to get something that looks like $dx[n]/dn$.

Solution:

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= e^{j\omega_0 n} - e^{j\omega_0(n-1)} \\ &= e^{j\omega_0 n}(1 - e^{-j\omega_0}) \\ &\approx j\omega_0 e^{-j\omega_0 n} \end{aligned}$$

which is $dx[n]/dn$.