ECE 401 Signal and Image Analysis
Homework 2

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: Wednesday, 9/8/2021; Due: Monday, 9/13/2021
Reading: *DSP First* pp. 12-34, 50-58, 61-71

Problem 2.1

Suppose that

$$v(t) = 2\cos(2\pi 880t) + 2\sin(2\pi 1320t)$$

What is the fundamental frequency? What are the Fourier series coefficients, V_k?

Solution: The fundamental frequency is the GCD of 880 and 1320, which is $F_0 = 440$Hz. The component at $k = 2$ ($f = 2F_0 = 880$) is

$$2\cos(2\pi 2F_0 t) = e^{j2\pi 2F_0 t} + e^{-j2\pi 2F_0 t}$$

so $V_2 = V_{-2} = 1$. The component at $k = 3$ ($f = 3F_0 = 1320$) is

$$2\sin(2\pi 3F_0 t) = 2\left(\frac{e^{j2\pi 3F_0 t} - e^{-j2\pi 3F_0 t}}{2j}\right)$$

so $V_3 = \frac{1}{j}$, and $V_{-3} = \frac{1}{-j}$. Putting it all together,

$$V_k = \begin{cases} 1 & k = 2, k = -2 \\ 1/j & k = 3 \\ -1/j & k = -3 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.2

Suppose that $x(t)$ is a square wave with a period of T_0, and with the following definition:

$$x(t) = \begin{cases} \frac{1}{2} & -\frac{T_0}{4} < t < \frac{T_0}{4} \\ -\frac{1}{2} & \frac{T_0}{4} < t < \frac{3T_0}{4} \end{cases}$$

We showed in class that the Fourier coefficients of this square wave are

$$X_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(|k|-1)/2}}{\pi k} & k \text{ odd} \end{cases}$$

Remember that the real-valued coefficients correspond to a Fourier series where all of the components are cosines. That makes sense, because the signal has even symmetry ($x(t) = x(-t)$), just like a cosine.

Suppose that we delay the signal by one quarter period, to produce the signal

$$y(t) = \begin{cases} \frac{1}{2} & 0 < t < \frac{T_0}{2} \\ -\frac{1}{2} & \frac{T_0}{2} < t < T_0 \end{cases}$$

Notice that $y(t)$ has odd symmetry ($y(t) = -y(-t)$), just like a sine wave. In that case, we might speculate that the Fourier seires expansion will be composed entirely of sine waves, i.e., the Fourier series coefficients, Y_k, will all be imaginary numbers.

Use the time-delay property of the spectrum (from lecture 3) to find out what happens to X_k when $x(t)$ is delayed by exactly one quarter period.

Solution: The time-delay property of a spectrum is that if $y(t) = x(t - \tau)$, then

$$Y_k = X_k e^{-j2\pi f_k \tau}$$

where, in our case, $f_k = kF_0 = k/T_0$, and $\tau = T_0/4$, so

$$Y_k = X_k e^{-j2\pi(k/T_0)(T_0/4)} = X_k e^{-j\pi k/2} = X_k \times (-j)^k$$

The multiplier $(-j)^k$ is imaginary for odd values of k, and real for even values of j. But we already know that $X_k = 0$ for even values of j, so this time delay will make all of its nonzero coefficients into imaginary numbers. Specifically,

$$Y_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2} j}{\pi k} & k \text{ odd} \end{cases}$$

Problem 2.3

Suppose that $z(t)$ is a triangle wave with a period of T_0, and with the following definition:

$$z(t) = \begin{cases} \frac{t}{2} - \frac{T_0}{4} & 0 < t < \frac{T_0}{2} \\ -\frac{t}{2} + \frac{3T_0}{4} & \frac{T_0}{2} < t < T_0 \end{cases}$$

Notice that this signal is exactly the anti-derivative of the signal $y(t)$ from problem (1), i.e., $y(t) = dz/dt$. Use the differentiation property of the spectrum (from lecture 3) in order to find the Fourier series coefficients Z_k.

Solution: The differentiation property of the spectrum says that, if $y(t) = dz/dt$, then

$$Y_k = j2\pi f_k Z_k$$

where, in our case, $f_k = kF_0 = k/T_0$, so $Y_k = j2\pi k Z_k / T_0$, so

$$Z_k = \frac{T_0 Y_k}{P} j2\pi k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2}}{2\pi^2 k^2} & k \text{ odd} \end{cases}$$

Notice that the Z_k are real numbers! If you plot $z(t)$ as a function of time, you can see that it has even symmetry ($z(t) = z(-t)$).

Problem 2.4

Suppose that a violin is playing the note A4 (440Hz), but our recording quality is bad, so we only get the first two harmonics:

$$x(t) = \sum_{k=-2}^{2} a_k e^{j2\pi k 440 t}$$

Suppose we measure the spectrum, and find it to be

$$\{(-880, 0.01), (-440, 1), (0, 0), (440, 1), (880, 0.01)\}$$

In order to improve the balance a little, we try differentiating the tone. Our differentiator also imposes a delay and a DC offset, though, so what we get is

$$y(t) = \frac{dx(t-0.001)}{dt} + 1.5$$

Find the spectrum of $y(t)$.

Solution: The time delay multiplies each coefficient by a phase offset term, $e^{-j2\pi f_k \tau}$. Differentiation multiplies each coefficient by $j2\pi f_k$. The DC offset just adds 1.5 to the zeroth spectral coefficient. The resulting spectrum is

$$\left\{\left(-880, -j2\pi 8.8 e^{j2\pi 0.88}\right), \left(-440, -j2\pi 440 e^{j2\pi 0.44}\right), (0, 1.5), \left(440, j2\pi 440 e^{-j2\pi 0.44}\right), \left(880, j2\pi 8.8 e^{-j2\pi 0.88}\right)\right\}$$

which can be written as

$$\left\{\left(-880, 2\pi 8.8 e^{j(2\pi 0.88-\frac{\pi}{2})}\right), \left(-440, 2\pi 440 e^{j(2\pi 0.44-\frac{\pi}{2})}\right), (0, 1.5), \left(440, 2\pi 440 e^{-j(2\pi 0.44-\frac{\pi}{2})}\right), \left(880, 2\pi 8.8 e^{-j(2\pi 0.88-\frac{\pi}{2})}\right)\right\}$$

or equivalently

$$\left\{\left(-880, 17.6\pi e^{j1.26\pi}\right), \left(-440, 880\pi e^{j0.38\pi 0.44}\right), (0, 1.5), \left(440, 880\pi e^{-j0.38\pi}\right), \left(880, 17.6\pi e^{-j1.26\pi}\right)\right\}$$