Lecture 8: Filtering Periodic Signals

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

What is Signal Processing, Really?

- When we process a signal, usually, we're trying to enhance the meaningful part, and reduce the noise.
- **Spectrum** helps us to understand which part is meaningful, and which part is noise.
- **Convolution** (a.k.a. filtering) is the tool we use to perform the enhancement.
- **Frequency Response** of a filter tells us exactly which frequencies it will enhance, and which it will reduce.

Review: Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

Review: DFT & Fourier Series

The most useful type of spectrum is a Fourier series (in discrete time: a DFT). **Any** periodic signal with a period of N samples, $x[n+N]=x[n]$, can be written as

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N}$$

which is a special case of the spectrum for periodic signals:

$$\omega_0=\frac{2\pi}{N}\frac{\text{radians}}{\text{sample}},\quad F_0=\frac{1}{T_0}\frac{\text{cycles}}{\text{second}},\quad T_0=\frac{N}{F_s}\frac{\text{seconds}}{\text{cycle}},\quad N=\frac{\text{samples}}{\text{cycle}}$$

and

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}$$

Frequency Response

- **Tones in $\rightarrow$ Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = G(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |G(\omega)|\cos(\omega n + \angle G(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|G(\omega)|\cos(\omega n + \theta + \angle G(\omega))$$

- where the **Frequency Response** is given by

$$G(\omega) = \sum_m g[m]e^{-j\omega m}$$

Outline

Delta: the do-nothing filter

First, let's define a do-nothing filter, called $\delta[n]$:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Its frequency response is

$$\sum_m \delta[m] e^{-j\omega m} = 1$$

This has the property that, when you convolve it within anything, you get that thing back again:

$$x[n] * \delta[n] = x[n]$$

Outline

A Pure-Delay "Filter"

One thing we can do to a signal is to just delay it, by n_0 samples:

$$y[n] = x[n - n_0]$$

Even this very simple operation can be written as a convolution:

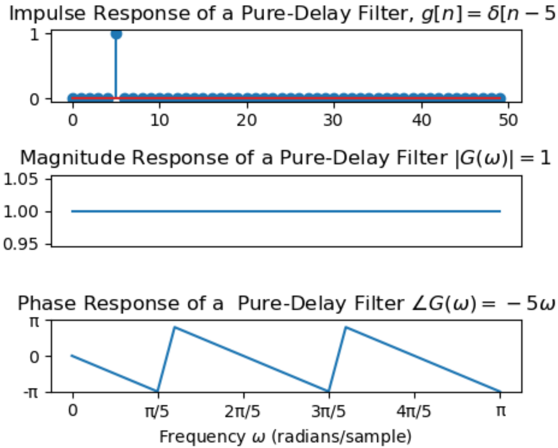
$$y[n] = g[n] * x[n]$$

where the "filter," $g[n]$, is just

$$g[n] = \delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$

Impulse Response of A Pure-Delay "Filter"

Here is the impulse response of a pure-delay "filter" (and the magnitude and phase responses, which we'll talk about next).



Frequency Response of A Pure-Delay "Filter"

$$g[n] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$

The frequency response is

$$G(\omega) = \sum_m g[m] e^{-j\omega m} = e^{-j\omega n_0}$$

Magnitude and Phase Response of A Pure-Delay "Filter"

$$G(\omega) = \sum_m g[m] e^{-j\omega m} = e^{-j\omega n_0}$$

Notice that the magnitude and phase response of this filter are

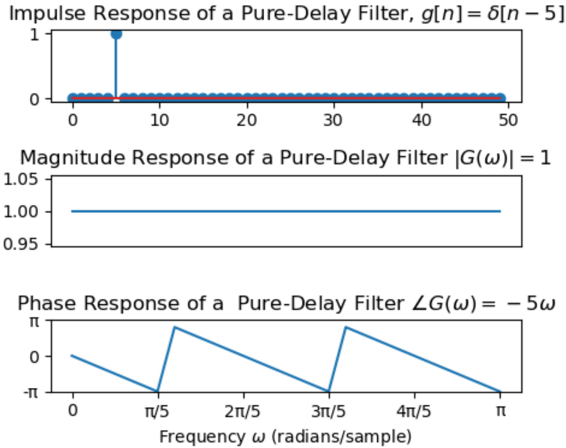
$$|G(\omega)| = 1$$
$$\angle G(\omega) = -\omega n_0$$

So, for example, if have an input of $x[n] = \cos(\omega n)$, the output would be

$$y[n] = |G(\omega)| \cos\left(\omega n + \angle G(\omega)\right) = \cos\left(\omega n - \omega n_0\right)$$

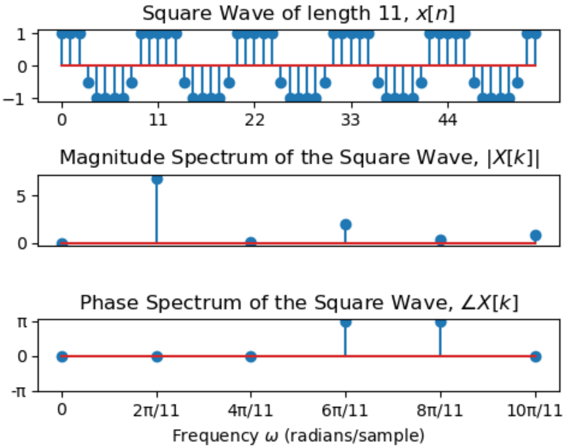
Magnitude and Phase Response of A Pure-Delay "Filter"

Here are the magnitude and phase response of the pure delay filter.

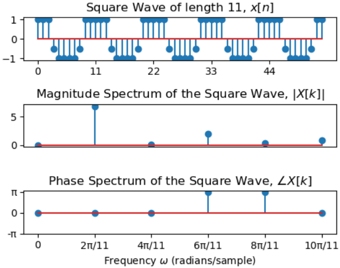


Spectrum of a Square Wave

Let's prove that the "pure delay" filter changes the phase spectrum, but has no influence on the magnitude spectrum. As the input, here's an (almost) square wave, with a period of 11 samples:



More about the phase spectrum

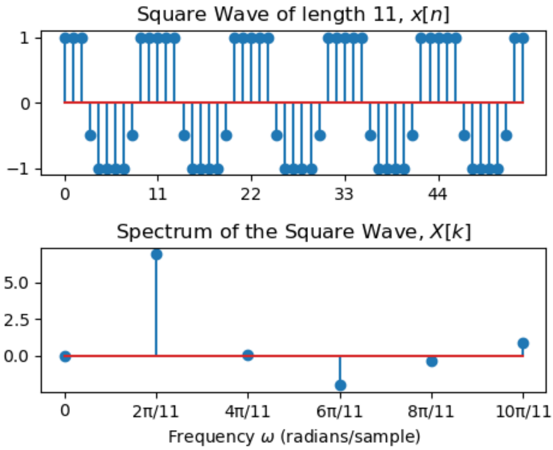


Notice that, for the phase spectrum of a square wave, the phase spectrum is either $\angle X[k] = 0$ or $\angle X[k] = \pi$. That means that the spectrum is real-valued, with no complex part:

- **Positive real:** $X[k] = |X[k]|$
- **Negative real:** $X[k] = -|X[k]| = |X[k]|e^{j\pi}$

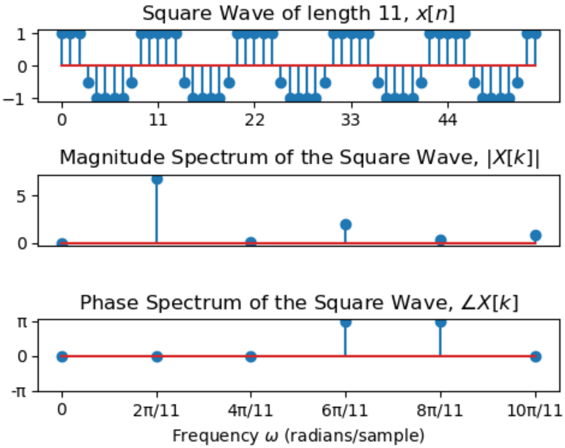
More about the phase spectrum

Having discovered that the square wave has a real-valued $X[k]$, we could just plot $X[k]$ itself, instead of plotting its magnitude and phase:



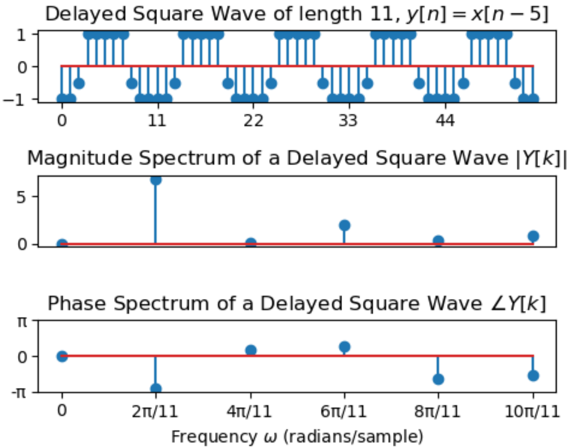
More about the phase spectrum

But delaying the signal to compute $y[n] = x[n-5]$ is going to change the phase, so $Y[k]$ won't be real-valued. In preparation for $Y[k]$, let's go back to plotting the magnitude and phase separately:



Spectrum: Delayed Square Wave

Anyway, here's the square wave, after being delayed by the pure-delay filter:



You can see that magnitude's unchanged, but phase is changed.

Outline

Output of a filter in response to a periodic signal

Suppose the input to a signal is periodic, $x[n+N] = x[n]$. That means the signal is made up of pure tones, at multiples of the fundamental:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jnk\omega_0}$$

Therefore, if we pass it through a filter,

$$y[n] = g[n] * x[n],$$

we get:

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{jnk\omega_0}, \quad Y[k] = G(k\omega_0)X[k]$$

Cascaded filters

Suppose I pass the signal through filter $g[n]$, then pass it through another filter, $f[n]$:

$$y[n] = f[n] * (g[n] * x[n]),$$

we get a signal $y[n]$ whose spectrum is:

$$Y[k] = F(k\omega_0)G(k\omega_0)X[k]$$

Convolution is commutative and associative

You know that multiplication is both commutative and associative. If $H(\omega) = F(\omega)G(\omega)$, then

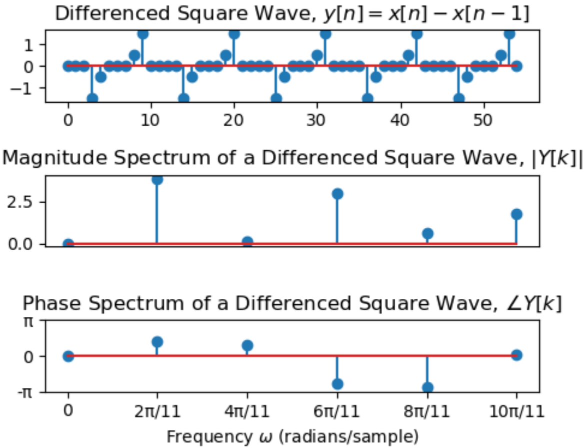
$$Y[k] = F(k\omega_0)G(k\omega_0)X[k] = G(k\omega_0)F(k\omega_0)X[k] = H(k\omega_0)X[k]$$

and therefore:

$$y[n] = f[n] * (g[n] * x[n]) = g[n] * (f[n] * x[n]) = h[n] * x[n]$$

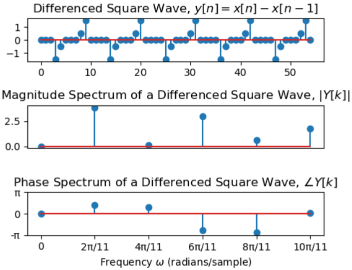
Example: Differenced Square Wave

Suppose we define $x[n]$ =square wave, $g[n]$ = pure-delay filter, and $f[n]$ = first-difference filter. Here's $f[n] * x[n]$:



Example: Differenced Square Wave

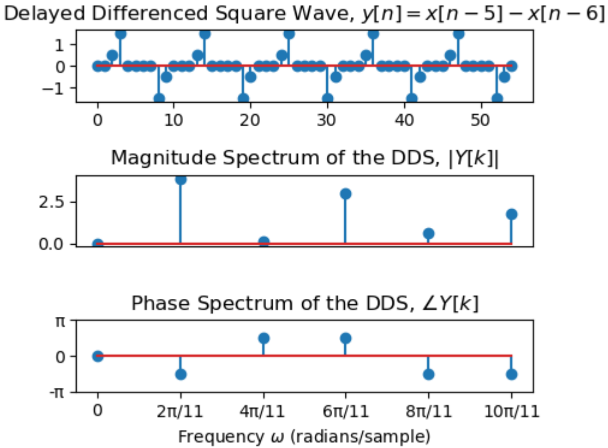
Suppose we define $x[n]$ =square wave, $g[n]$ = pure-delay filter, and $f[n]$ = first-difference filter. Here's $f[n] * x[n]$:



You can see that the differencing operation has raised the amplitude of the higher harmonics, because the first-difference filter is a high-pass filter, as you saw last time.

Example: Delayed Differenced Square Wave

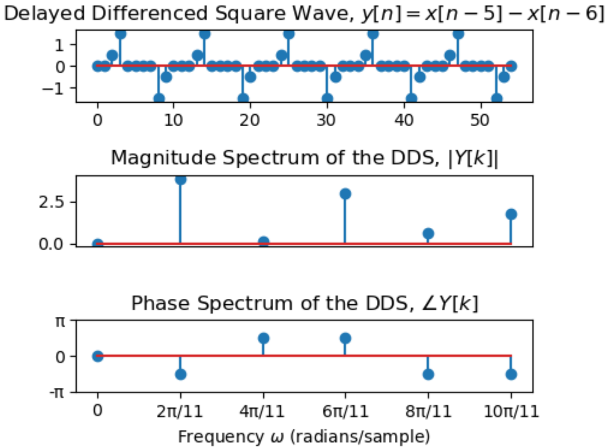
Here's $g[n] * f[n] * x[n]$, the delayed differenced square wave:



Again, the delay operation has changed the phases, but not the magnitudes.

Example: Differenced Delayed Square Wave

Here's $f[n] * g[n] * x[n]$, the differenced delayed square wave (hint: it's exactly the same as the previous slide!!)



Magnitude and Phase of Cascaded Frequency Responses

Notice, in the previous two slides, that the magnitudes multiply:

$$|H(\omega)| = |F(\omega)||G(\omega)|,$$

but the phases add:

$$\angle H(\omega) = \angle F(\omega) + \angle G(\omega).$$

That's because:

$$F(\omega)G(\omega) = |F(\omega)|e^{j\angle F(\omega)}|G(\omega)|e^{j\angle G(\omega)} = |F(\omega)||G(\omega)|e^{j(\angle F(\omega)+\angle G(\omega))}$$

Outline

Local Average Filters

Let's go back to the local averaging filter from Lecture 2. I want to define two different types of local average: centered, and delayed.

- **Centered local average:** This one averages $\left(\frac{L-1}{2}\right)$ future samples, $\left(\frac{L-1}{2}\right)$ past samples, and $x[n]$:

$$y_c[n] = \frac{1}{L} \sum_{m=-\left(\frac{L-1}{2}\right)}^{\left(\frac{L-1}{2}\right)} x[n-m]$$

- **Delayed local average:** This one averages $x[n]$ and $L-1$ of its past samples:

$$y_d[n] = \frac{1}{L} \sum_{m=0}^{L-1} x[n-m]$$

Notice that $y_d[n] = y_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

Local Average Filters

We can write both of these as filters:

- **Centered local average:**

$$y_c[n] = f_c[n] * x[n]$$

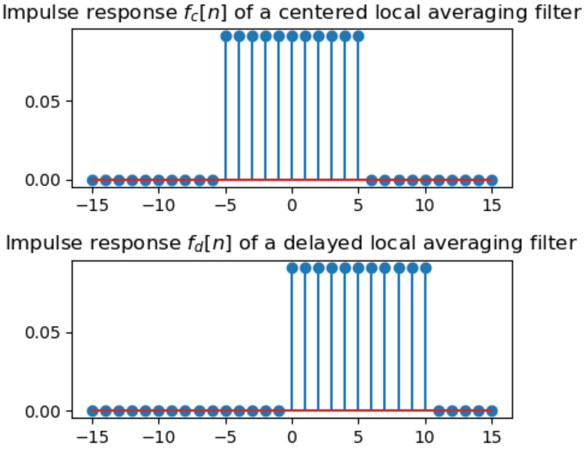
$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

- **Delayed local average:**

$$y_d[n] = f_d[n] * x[n]$$

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Local Average Filters



Notice that $f_d[n] = f_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

The relationship between centered local average and delayed local average

Suppose we define our pure delay filter,

$$g[n] = \delta\left[n - \frac{L-1}{2}\right] = \begin{cases} 1 & n = \frac{L-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Using $g[n]$, here are lots of different ways we can write the relationship between $y_d[n]$, $y_c[n]$, and $x[n]$:

$$\begin{aligned} y_d[n] &= f_d[n] * x[n] \\ y_c[n] &= f_c[n] * x[n] \\ y_d[n] &= g[n] * y_c[n] = g[n] * f_c[n] * x[n] \\ f_d[n] &= g[n] * f_c[n] \end{aligned}$$

The relationship between centered local average and delayed local average

Remember the frequency response of a pure delay filter:

$$G(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)}$$

We have not yet figured out what $F_c(\omega)$ and $F_d(\omega)$ are. But whatever they are, we know that

$$f_d[n] = g[n] * f_c[n]$$

and therefore

$$F_d(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)} F_c(\omega)$$

The frequency response of a local average filter

Let's find the frequency response of

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

The formula is

$$F_d(\omega) = \sum_m f[m] e^{-j\omega m},$$

so,

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

The frequency response of a local average filter

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

This is just a standard geometric series,

$$\sum_{m=0}^{L-1} a^m = \frac{1 - a^L}{1 - a},$$

so:

$$F_d(\omega) = \frac{1}{L} \left(\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} \right)$$

The frequency response of a local average filter

We now have an extremely useful transform pair:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right)$$

Let's attempt to convert that into polar form, so we can find magnitude and phase response. Notice that both the numerator and the denominator are subtractions of complex numbers, so we might be able to use $2j\sin(x) = e^{jx} - e^{-jx}$ for some x. Let's try:

$$\begin{aligned}
\frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right) &= \frac{1}{L}\frac{e^{-j\omega L/2}}{e^{-j\omega/2}}\left(\frac{e^{j\omega L/2}-e^{-j\omega L/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\
&= e^{-j\omega\left(\frac{L-1}{2}\right)}\frac{1}{L}\left(\frac{2j\sin(\omega L/2)}{2j\sin(\omega/2)}\right) \\
&= e^{-j\omega\left(\frac{L-1}{2}\right)}\frac{1}{L}\left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right)
\end{aligned}$$

The frequency response of a local average filter

Now we have $F_d(\omega)$ in almost magnitude-phase form:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \left(\frac{\sin(\omega L/2)}{L\sin(\omega/2)}\right) e^{-j\omega\left(\frac{L-1}{2}\right)}$$

By the way, remember we discovered that

$$f_d[n] = g[n] * f_c[n] \quad \leftrightarrow \quad F_d(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)} F_c(\omega)$$

Notice anything?

Dirichlet form

The textbook calls this function the "Dirichlet form:"

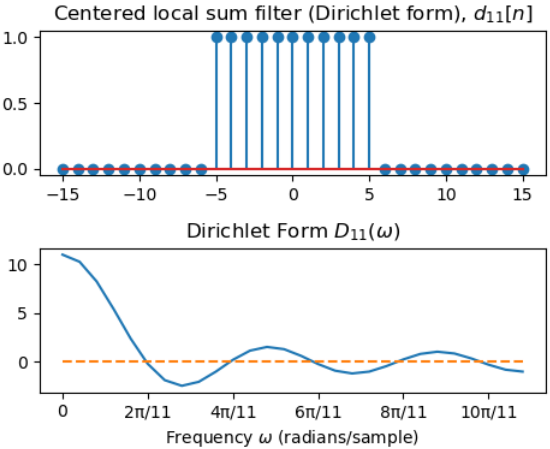
$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

That is, exactly, the frequency response of a centered local sum filter:

$$d_L[n] = \begin{cases} 1 & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Dirichlet form

Here's what it looks like:



Dirichlet form

Since every local averaging filter is based on Dirichlet form, it's worth spending some time to understand it better.

$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

- It's equal to zero every time $\omega L/2$ is a multiple of π. So

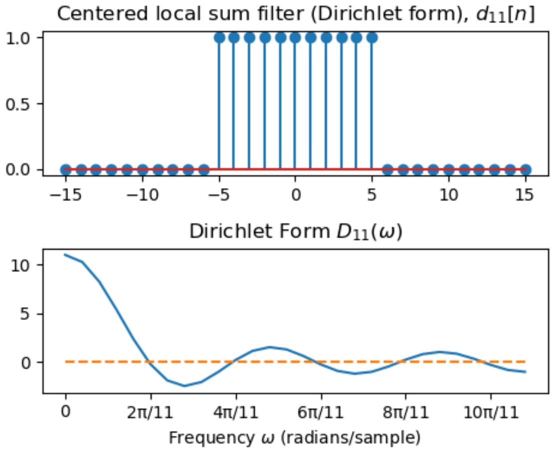
$$D_L\left(\frac{2\pi k}{L}\right) = 0 \text{ for all integers } k \text{ except } k = 0$$

- At $\omega = 0$, the value of $\frac{\sin(\omega L/2)}{\sin(\omega/2)}$ is undefined, but it's posssible to prove that $\lim_{\omega\to 0} D_L(\omega) = L$. To make life easy, we'll just define it that way:

$$\text{DEFINE: } D_L(0) = L$$

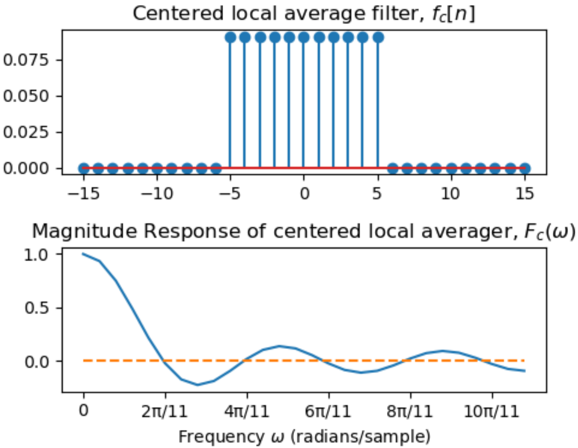
Dirichlet form

Here's what it looks like:



Local averaging filter

Here's what the centered local averaging filter looks like. Notice that it's just $1/L$ times the Dirichlet form:



Outline

Output of a filter in response to a periodic signal

Suppose the input to a signal is periodic, $x[n+N] = x[n]$. That means the signal is made up of pure tones, at multiples of the fundamental:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jnk\omega_0}$$

Therefore, if we pass it through a filter,

$$y[n] = g[n] * x[n],$$

we get:

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{jnk\omega_0}, \quad Y[k] = G\left(\frac{2\pi k}{N}\right) X[k]$$

Output of a local averaging filter in response to a periodic signal

What if we use a local averaging filter, averaging over one complete period of $x[n]$?

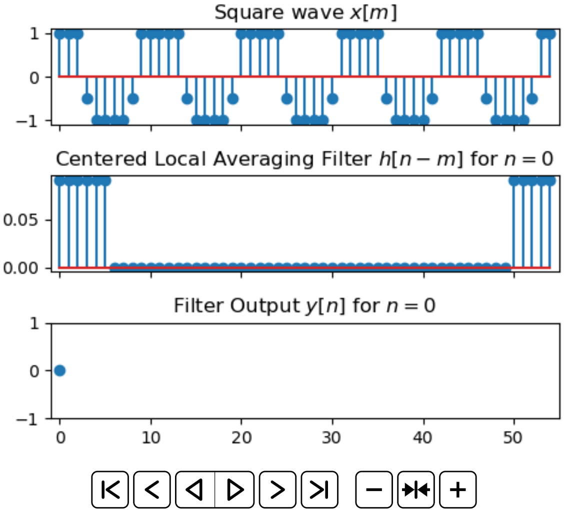
$$y[n] = f_c[n] * x[n]$$

$$f_c[n] = \begin{cases} \frac{1}{N} & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_c(\omega) = \frac{\sin(\omega N/2)}{N \sin(\omega/2)}$$

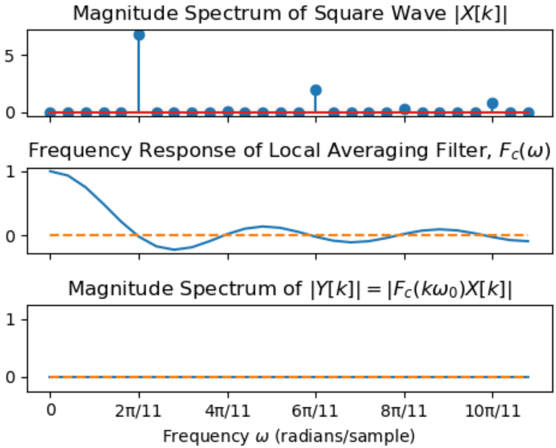
We get:

$$Y[k] = F_c\left(\frac{2\pi k}{N}\right) X[k] = 0!!!$$

Local averaging of a periodic signal: Time domain



Local averaging of a periodic signal: Frequency domain

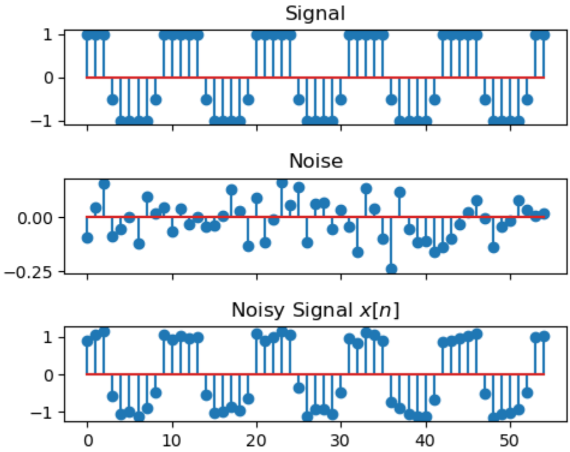


Local averaging of a periodic signal

- If you compute the average of a periodic signal, over one complete period the average is zero (assuming that there is no DC offset).
- What's left in $y[n] = f_c[n] * x[n]$ is just the background noise.

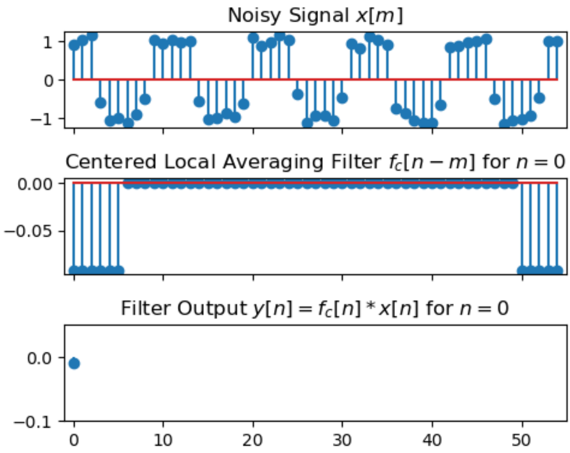
A noisy signal

Here's a noisy signal:



A noise-only signal!

Here's a noisy signal, averaged over each period. Notice, all that's left in $y[n]$ is the noise!



Local averaging of a periodic signal

- If you compute the average of a periodic signal, over one complete period the average is zero (assuming that there is no DC offset).
- What's left in $y[n] = f_c[n] * x[n]$ is just the background noise.
- Can we subtract the background noise from the original signal?

$$z[n] = x[n] - y[n]$$

  Would that give us a version of $x[n]$ with less background noise?

Local averaging of a periodic signal

- The noisy signal is

$$x[n] = \delta[n] * x[n]$$

- Noise is all that's left after we get rid of the periodic part, using

$$y[n] = f_c[n] * x[n]$$

- Can we subtract the background noise from the original signal?

$$z[n] = x[n] - y[n] = (\delta[n] - f_c[n]) * x[n]$$

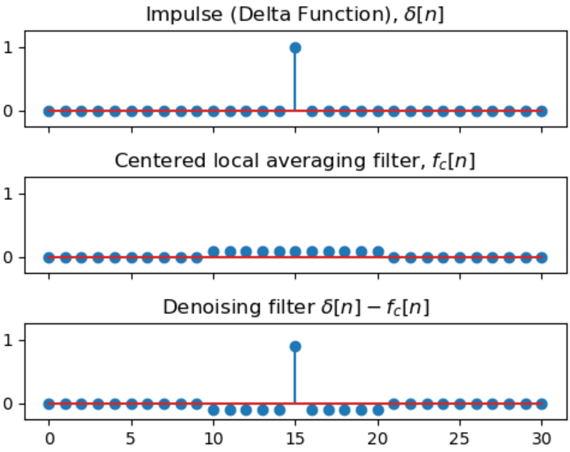
A denoising filter

So the de-noising filter is:

$$h[n] = \delta[n] - f_c[n]$$

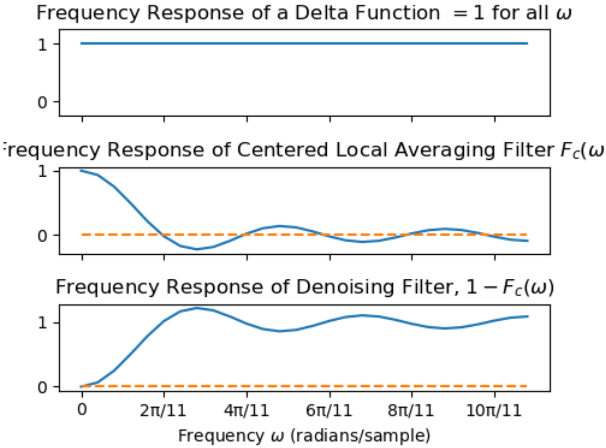
Denoising filter

Here are the delta function, local averaging filter, and denoising filter in the time domain:



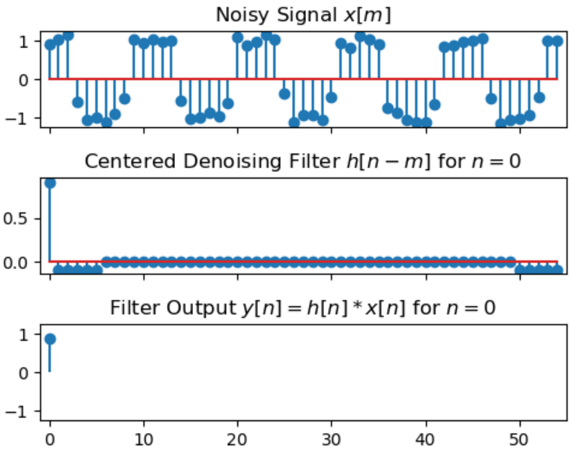
Denoising filter

Here are the delta function, local averaging filter, and denoising filter in the frequency domain:



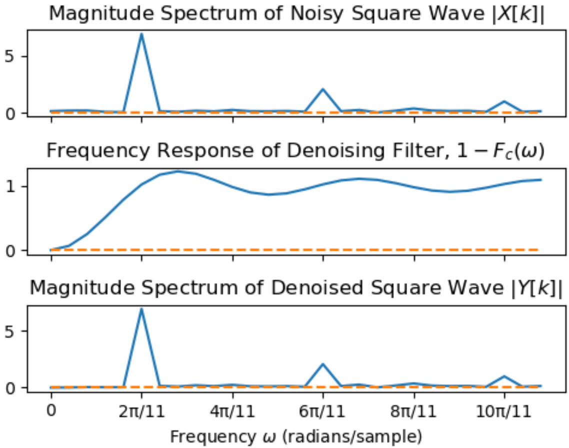
A denoised signal

Here's a denoised signal:



A denoised signal

Here's the noisy and denoised signals in the frequency domain:



Not a very good denoising filter?

OK, so that wasn't really a very **good** denoising filter! Can we do better? . . . wait until after the midterm, and we'll talk about how to do better. . .

Outline

Summary

- The **Pure Delay Filter** has $|G(\omega)| = 1$, $\angle G(\omega) = -\omega n_0$:

$$g[m] = \delta[n - n_0] \quad \leftrightarrow \quad G(\omega) = e^{-j\omega n_0}$$

- **Cascaded LTI Systems** convolve their impulse responses, equivalently, they multiply their frequency responses:

$$h[n] = f[n] * g[n] \quad \leftrightarrow \quad H(\omega) = F(\omega)G(\omega)$$

- The **Centered Local Averaging Filter** is $1/L$ times the Dirichlet form:

$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_c(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)}$$

- The **Delayed Local Averaging Filter** is $f_c[n]$, delayed by half of its length:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)} e^{-j\omega\left(\frac{L-1}{2}\right)}$$