ECE 361: Lecture 5: $M$-ary Amplitude Shift Keying

5.1. The Signals in $M$-ary Amplitude Shift Keying

In $M$-ary Amplitude Shift Keying ($M$-ASK), one of $M$ distinct equally likely messages is conveyed during each channel use. Of course, in order to avoid trivialities, we assume that $M$ is 2 or more. The basic signal in $M$-ary ASK is a unit-energy signal $\psi(t)$, and the $M$ possible transmitted signals all are different multiples of $\psi(t)$. Thus, different amounts of energy are transmitted depending on which of the signals is transmitted. The signals are as follows:

- If $M$ is even, the signals are $\pm(M - 1)\sqrt{E}\psi(t)$, $\pm(M - 3)\sqrt{E}\psi(t)$, ..., $\pm3\sqrt{E}\psi(t)$, $\pm\sqrt{E}\psi(t)$.
- If $M$ is odd, the signals are $\pm(M - 1)\sqrt{E}\psi(t)$, $\pm(M - 3)\sqrt{E}\psi(t)$, ..., $\pm4\sqrt{E}\psi(t)$, $\pm2\sqrt{E}\psi(t)$, 0.

Notice that in both cases, the maximum energy is $E_{\text{max}} = (M - 1)^2E$, while the average energy can be shown to be $\bar{E} = M^2 - 1 \frac{E}{3}$. The ratio of the maximum to the average is $\frac{E_{\text{max}}}{\bar{E}} = 3\frac{M - 1}{M + 1}$, and thus the maximum transmitted energy is nearly three times the average transmitted energy. Since $\log_2 M$ bits are conveyed in each transmission, the data rate is $R = \log_2 M$ bits per transmission, and the average energy per bit is $E_b = \frac{\bar{E}}{\log_2 M} = \frac{\bar{E}}{R}$.

5.2. The Receiver and Error Probabilities

Assuming that $M$-ASK system is operating on an additive white Gaussian noise channel, the receiver consists of a linear filter matched to $\psi(t)$, a sampler, and a decision device. Taking $h(t) = \psi(-t)$ for convenience and remembering that $\psi(t)$ is a unit energy signal, the noise variance at the output is $\frac{N_0}{2}$ while the signal levels in the sample output are just $\pm(M - 1)\sqrt{E}, \pm(M - 3)\sqrt{E},$ ..., and thus spaced $2\sqrt{E}$ apart. The decision device has thresholds set at $\pm(M - 2)\sqrt{E}, \pm(M - 2)\sqrt{E},$ ..., which are also spaced $2\sqrt{E}$ apart. Note that the smallest thresholds are set at $\pm\sqrt{E}$ if $M$ is odd, while if $M$ is even, there is a threshold at 0 and also thresholds at $\pm2\sqrt{E}$.

The average symbol error probability $P(E)$ is the probability that the receiver decides that the wrong message was sent. The conditional symbol error probability (conditioned on which message was transmitted) depends on whether the message sent corresponds to an exterior point, that is, an amplitude of $\pm(M - 1)\sqrt{E}$ at the output of the filter, or an interior point (any of the other $M - 2$ points). If an exterior point, say $-(M - 1)\sqrt{E}$ is transmitted, the filter output $Y$ is a $\mathcal{N}(-(M - 1)\sqrt{E}, \frac{N_0}{2})$ random variable, and a symbol error occurs if the event $\{Y > -(M - 2)\sqrt{E}\}$ occurs. It is easy to verify that the error probability is $Q\left(\sqrt{\frac{2E}{N_0}}\right)$. On the other hand, if the transmitted message corresponds to an interior point, then $Y$ is a $\mathcal{N}(K\sqrt{E}, \frac{N_0}{2})$ random variable, and a symbol error occurs if either the event $\{Y > (K + 1)\sqrt{E}\}$ or the event $\{Y < (K - 1)\sqrt{E}\}$ occurs. The symbol error probability is thus $2Q\left(\sqrt{\frac{2E}{N_0}}\right)$. The law of total probability thus gives

$$P(E) = \frac{1}{M} \left[ 2 \cdot Q\left(\sqrt{\frac{2E}{N_0}}\right) + (M - 2) \cdot 2Q\left(\sqrt{\frac{2E}{N_0}}\right) \right] = 2\left(\frac{M - 1}{M}\right)Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$$= 2\left(1 - \frac{1}{M}\right)Q\left(\sqrt{\frac{2E}{N_0}}\right) \approx 2Q\left(\sqrt{\frac{2E}{N_0}}\right) \text{ for large } M.$$

\(^{1}\)See also Section 2 of Lecture 1.
Notice that for $M = 2$, we have binary antipodal signaling and the formula correctly gives us $P(E) = Q(\sqrt{2E/N_0})$. For large $M$ though, the average symbol error probability is nearly twice the error probability for binary antipodal signaling for the same value of $E$. Indeed, since $Q(\cdot)$ is a rapidly decreasing function of its argument, a very small increase in energy could easily compensate for the extra factor of 2. This facile comparison might mislead us into thinking that large amounts of data can easily be transmitted via $M$-ASK with the same error probability as binary antipodal signaling. Unfortunately, the price to be paid to achieve error probability $2Q(\sqrt{E/N_0})$ is prohibitive at high data rates. Let us re-write several of the important parameters to get a better perspective on $M$-ASK systems.

- Suppose that $E$ is fixed, and is such that the value of $P(E) \approx 2Q(\sqrt{E/N_0})$ is quite satisfactory. What data rates $R$ are feasible?

$$P(E) = 2 \left(1 - \frac{1}{M}\right) Q\left(\sqrt{\frac{2E}{N_0}}\right) = 2 \left(1 - \frac{1}{M}\right) Q\left(\sqrt{\frac{6E}{(M^2 - 1)N_0}}\right)$$

Now as $M$ increases, the argument of $Q(\cdot)$ decreases rapidly towards 0 and thus the function value approaches 1/2. Thus, the average symbol error probability approaches $(M - 1)/M$ as $M$ increases. This is exactly the error probability that we would get if the receiver ignored the received signal entirely and simply chose among the $M$ messages at random! Thus, for the realistic situation of a constraint on the average or maximum energy transmitted, the data rate cannot be increased unduly without making the system worthless. For this reason, values of $M$ larger than 8 (or occasionally 16) are not used in $M$-ASK systems.

### 5.3. Bit Error Probabilities

We have seen the behavior of the average symbol error probability in the previous section. We now turn to considering bit error probabilities. Let us assume that $M = 2^m$ where $m \geq 2$ is an integer, so that each use of the channel sends $m$ bits which we shall refer to as a nybble. Which nybble should we associate with each of the $M$ signals described in Section 1 above? The answer is best developed by considering what happens in the receiver.

Let $a$ denote the nybble associated with the transmitted signal $K\sqrt{E}\psi(t)$ which we assume, as in Section 2, to correspond to an interior point. Thus, the filter output is $Y$ is a $\mathcal{N}(K\sqrt{E}, \frac{N_0}{2})$ random variable. The thresholds are set at $(K \pm 1)\sqrt{E}, (K \pm 3)\sqrt{E}, (K \pm 5)\sqrt{E} \ldots$. The receiver

- makes a correct decision if $(K - 1)\sqrt{E} < Y < (K + 1)\sqrt{E}$. The probability of this is

$$1 - P(E) = 1 - 2Q(\sqrt{E/N_0})$$

- makes the incorrect decision that the neighboring signal $(K + 2)\sqrt{E}\psi(t)$ (with associated nybble $a^+$) was transmitted if $(K + 1)\sqrt{E} < Y < (K + 3)\sqrt{E}$. The probability of this is

$$Q(\sqrt{2E/N_0}) - Q(3\sqrt{2E/N_0}) \approx Q(\sqrt{2E/N_0})$$

since $Q(\cdot)$ is a rapidly decreasing function of its argument.
• makes the incorrect decision that the next neighboring signal \((K + 4)\sqrt{E}\psi(t)\) with associated nybble \(a^+\) was transmitted if \((K + 3)\sqrt{E} < \gamma < (K + 5)\sqrt{E}\). The probability of this is

\[
Q(3\sqrt{2E}/N_0) - Q(5\sqrt{2E}/N_0) \approx Q(\sqrt{2E}/N_0) - Q(3\sqrt{2E}/N_0).
\]

Note also that

\[
Q(3\sqrt{2E}/N_0) - Q(5\sqrt{2E}/N_0) \approx Q(3\sqrt{2E}/N_0).
\]

• makes the incorrect decision that the neighboring signal \((K - 2)\sqrt{E}\psi(t)\) with associated nybble \(a^-\) was transmitted if \((K - 3)\sqrt{E} < \gamma < (K - 1)\sqrt{E}\). The probability of this is also

\[
Q(\sqrt{2E}/N_0) - Q(3\sqrt{2E}/N_0) \approx Q(\sqrt{2E}/N_0).
\]

• makes the incorrect decision that the next neighboring signal \((K - 4)\sqrt{E}\psi(t)\) with associated nybble \(a^-\) was transmitted if \((K - 5)\sqrt{E} < \gamma < (K - 3)\sqrt{E}\). The probability of this is

\[
Q(3\sqrt{2E}/N_0) - Q(5\sqrt{2E}/N_0) \approx Q(\sqrt{2E}/N_0) - Q(3\sqrt{2E}/N_0).
\]

• . . . and so on . . .

The conclusion is that when the receiver makes an incorrect decision, the overwhelming probability is that it chooses a nearest neighbor of the correct signal, and it hardly ever chooses signals further away. Thus, an obvious way to minimize bit errors is to arrange matters such that nybbles \(a\) and \(a^+\) differ in one bit only, and similarly nybbles \(a\) and \(a^-\) differ in one bit only. Thus, going from signal \(-(M - 1)\sqrt{E}\psi(t)\) to signal \(+ (M - 1)\sqrt{E}\psi(t)\), we associate nybbles in Gray code order with signals so that nybbles associated with neighboring signals differ in only one bit. Thus, with probability very nearly \(P(E)\) only one of the \(m\) transmitted bits is in error, and with very small probability are two or more bits in error. Thus, the bit error probability is very nearly \(m^{-1}P(E)\). Think of the result in the following way. Consider the transmission of, say, \(10^9m\) data bits as \(10^9\) \(m\)-bit nybbles. Roughly \(10^9P(E)\) of these nybbles will be received in error, and most of them will have a single bit in error. Thus, the bit error rate is

\[
\text{Bit error rate} = \frac{10^9P(E) \text{ incorrect received bits}}{10^9m \text{ total received bits}} = \frac{P(E)}{m}
\]
as claimed.

Bit error probabilities in M-ASK systems are messy to deal with. Let \(B_i\) denote the event that the \(i\)-th bit of the nybble is in error. Then, \(P(B_1), P(B_2), \ldots, P(B_m)\) are usually different from one another, as are the conditional probabilities of these events conditioned on a specific signal being transmitted. Also, the events \(B_1, B_2, \ldots, B_m\) are not independent events (nor are they conditionally independent given the transmitted signal). Usually, it is best to ignore the fine details and deal with the average bit error probability \(P_b = (P(B_1) + P(B_2) + \cdots + P(B_m))/m\) (or the average bit error probability \(P_{b,i}\) conditioned on the \(i\)-th signal having been transmitted). Then, even in the absence of Gray coding, it is true that the bit error probabilities satisfy

\[
\frac{1}{m} P(E) \leq P_b \leq P(E)
\]

\[
\frac{1}{m} P(E_i) \leq P_{b,i} \leq P(E_i)
\]

where \(P_{\cdot, \cdot}\) denotes the conditional probability of symbol error given that the \(i\)-th signal has been transmitted. Thus, reducing the average symbol probability reduces the average bit error probability without having to worry about the fine details. As a last comment, we note that we saw that for fixed average transmitted energy, the symbol error probability approaches \((M - 1)/M\) as \(M \to \infty\). The average bit error rate is

\[
P_b \approx \frac{P(E)}{\log_2 M} = \frac{2}{\log_2 M} \frac{(M - 1)}{M} Q\left(\sqrt{\frac{6E}{(M^2 - 1)N_0}}\right)
\]

which seems to be approaching 0 since \(P(E) \to 1\) as \(M \to \infty\). and we have a factor of \(\log_2 M \to \infty\) in the denominator. However, this too is a false hope, and a careful analysis of the behavior of \(P_b\) as \(M \to \infty\) shows that \(P_b \to \frac{1}{2}\) as \(M \to \infty\), proving yet again that there is no such thing as a free lunch!