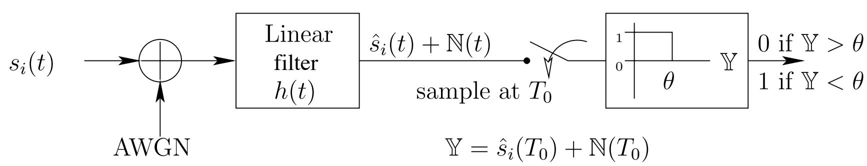
ECE 361: Lecture 4: Matched Filters – Part II

4.1 Introduction

In Lecture 3, we studied the the digital communications receiver model shown below where the signal $s_i(t)$, denoting one of two equally likely possible received signals $s_0(t)$ and $s_1(t)$ is processed through a linear time-invariant filter with impulse response $h(t)$ and then sampled at time T_0. The received signal is corrupted by additive white Gaussian noise (with two-sided power spectral density $N_0/2$) which also passes through the filter and corrupts the sample value which is thus $\mathbb{Y} = \hat{s}_i(T_0) + \mathbb{N}(T_0)$, where it is assumed that $\hat{s}_0(T_0) > \hat{s}_1(T_0)$. Conditioned on $s_i(t)$ being transmitted, $\mathbb{Y}$ is a Gaussian random variable with mean $\hat{s}_i(T_0)$ and variance $\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |h(t)|^2\,dt = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df$ where $H(f)$, the Fourier transform of $h(t)$, is the transfer function of the linear filter. The sample value $\mathbb{Y}$ is compared to the minimax threshold $\theta = \frac{1}{2}(\hat{s}_0(T_0) + \hat{s}_1(T_0))$ in the *decision device* whose input-output characteristic is as shown in the figure. Thus, the receiver decides that a 0 or a 1 was transmitted according as $\mathbb{Y}$ is larger than or smaller than θ. The error probability P_e achieved by this decision rule is $Q\left(\frac{\hat{s}_0(T_0)-\hat{s}_1(T_0)}{2\sigma}\right) = Q(\mathsf{SNR})$ where SNR (*signal-to-noise ratio*) is the value of the argument of the $Q(\cdot)$ function.



Since $Q(\cdot)$ is a decreasing function of its argument, , we can minimize P_e for any given filter by choosing the sampling time T_0 to be the time instant where $|\hat{s}_0(t) - \hat{s}_1(t)|$ has *maximum* value.[1] If $\hat{s}_0(T_0) > \hat{s}_1(T_0)$ as was assumed above, then the receiver shown above works just fine. But if $\hat{s}_0(T_0) < \hat{s}_1(T_0)$, then the decision rule shown in the figure above should be complemented: the receiver decides that a 0 or a 1 was transmitted according as $\mathbb{Y}$ is *smaller* than or *larger* than θ. It is usually easier to simply include an inverter in the filter, effectively changing $h(t)$ to $-h(t)$, or a logic inverter (NOT gate) at the output of the decision device than to redesign the decision device. Assuming that such a modification has been done to accommodate the possibility that $\hat{s}_0(T_0) < \hat{s}_1(T_0)$, the smallest error probability that can be achieved by using the given filter (with impulse response $h(t)$) and the given signals $s_0(t)$ and $s_1(t)$ is $Q\left(\frac{|\hat{s}_0(T_0)-\hat{s}_1(T_0)|}{2\sigma}\right) = Q(\mathsf{SNR})$. In other words, the maximum signal-to-noise ratio that the given filter can achieve for the given signals is $\mathsf{SNR} = \frac{|\hat{s}_0(T_0)-\hat{s}_1(T_0)|}{2\sigma}$. Other choices of sampling time at which the *signal separation* $|\hat{s}_0(t) - \hat{s}_1(t)|$ of the two filter outputs is smaller have smaller SNR and larger error probability.

In Lecture 3, we also found that for given signals $s_0(t)$ and $s_1(t)$, regardless of what filter, what sampling time, and what threshold the receiver uses, SNR cannot exceed the upper bound SNR^* shown in (4.1) below and the error probability cannot be smaller than the lower bound P_e^* shown in (4.2) below:

$$\mathsf{SNR} \le \mathsf{SNR}^* = \sqrt{\frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle}{2N_0}}, \tag{4.1}$$

$$P_e \ge P_e^* \quad = Q\left(\sqrt{\frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle}{2N_0}}\right). \tag{4.2}$$

Here $\mathcal{E}_0$ and $\mathcal{E}_1$ respectively denote the *energy* in the given received signals $s_0(t)$ and $s_1(t)$ and $\langle s_0, s_1\rangle$ denotes their *inner product*. Thus, $\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle$ is the energy in the *difference signal* $s(t) = s_0(t) - s_1(t)$,

[1] If there is more that one choice for T_0, any one of them can be taken to be T_0. Usually the earliest such time is the preferred choice.

and SNR^* and P_e^* depend on the signals $s_0(t)$ and $s_1(t)$ only through the energy of the difference signal $s(t) = s_0(t) - s_1(t)$. The exact *signal shapes and frequencies* of $s_0(t)$ and $s_1(t)$ do not matter except insofar as they affect the energy of $s(t)$. We also found that if *we choose* the sampling time T_0 to be some convenient instant, then a filter with impulse response $h(t) = \lambda[s_0(T_0 - t) - s_1(T_0 - t)] = \lambda s(T_0 - t)$ (where λ is some nonzero constant) *maximizes the signal separation* $|\hat{s}_0(T_0) - \hat{s}_1(T_0)|$ at *our chosen sampling time* T_0 and thus achieves the maximum signal-to-noise ratio SNR^* and minimum error probability P_e^*. If λ is a positive number, then $\hat{s}_0(T_0) > \hat{s}_1(T_0)$ and the receiver shown above works; else we have to fiddle with changing $h(t)$ to $-h(t)$ which is equivalent to replacing $\lambda < 0$ by $-\lambda = |\lambda| > 0$; that is, choosing the gain to be a positive number is what we should have been doing in the first place! This is the *optimum receiver* (in the sense of achieving the minimum possible error probability) for signals $s_0(t)$ and $s_1(t)$ in additive white Gaussian noise.[2] The optimum receiver for signals in additive white Gaussian noise is a *linear* receiver: filtering, sampling, and comparing to a threshold is all that is required.

Definition: A filter with impulse response $\lambda[s_0(T_0 - t) - s_1(T_0 - t)] = \lambda s(t)$, $\lambda > 0$, is said to be *matched* to the signals $s_0(t)$ and $s_1(t)$ at time T_0, or to be a *matched filter* for the signals $s_0(t)$ and $s_1(t)$ for sampling time T_0.

Note that the specification of the optimum sampling time is built into the definition of matched filter above. *We* choose a convenient sampling time T_0 and the definition of the matched filter for sampling time T_0 guarantees maximum SNR, that is, that the signal separation $\hat{s}_0(T_0) - \hat{s}_1(T_0)$ at our chosen sampling time T_0 is as large as possible (and positive too!) Some authorities reserve the name matched filter for the filter with impulse response $h_m(t) = s_0(-t) - s_1(-t)$, which corresponds to the special case $\lambda = 1$ and $T_0 = 0$ in the definition above. We call this the *canonical* matched filter and note that a non-canonical matched filter (that maximizes SNR at time T_0 as described in the boxed text above) is just the canonical matched filter (which maximizes SNR at $t = 0$) followed by a pure delay of T_0 to make this maximum occur at T_0 instead of at 0, and a pure gain of $\lambda > 0$. The canonical matched filter is certainly very convenient for analysis purposes since we don't have to carry around the parameters T_0 and λ, but it should always be remembered that canonical matched filters are generally *noncausal* filters. Indeed, if $s_0(t)$ and $s_1(t)$ have support $[0, T)$ (say), then any filter matched to $s_0(t)$ and $s_1(t)$ at any sampling time $T_0 < T$ will be a noncausal filter.

4.2 The matched filter in the time domain

For the canonical matched filter receiver where the filter impulse response is $h_m(t) = s_0(-t) - s_1(-t)$ and the sampling time is 0, the signal outputs $\hat{s}_0(0)$ and $\hat{s}_1(0)$ and the noise variance σ^2 at the sampling instant 0 are

$$\hat{s}_0(0) = \int_{-\infty}^{\infty} h_m(0-t)s_0(t)\,dt = \int_{-\infty}^{\infty} [s_0(t) - s_1(t)]s_0(t)\,dt = \mathcal{E}_0 - \langle s_0, s_1 \rangle \tag{4.3}$$

$$\hat{s}_1(0) = \int_{-\infty}^{\infty} h_m(0-t)s_1(t)\,dt = \int_{-\infty}^{\infty} [s_0(t) - s_1(t)]s_1(t)\,dt = \langle s_0, s_1 \rangle - \mathcal{E}_1 \tag{4.4}$$

$$\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |h(t)|^2\,dt \qquad = \frac{N_0}{2}\int_{-\infty}^{\infty} [s_0(t) - s_1(t)]^2\,dt = \frac{N_0}{2}\left[\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle\right] \tag{4.5}$$

Consequently,

$$\hat{s}_0(0) - \hat{s}_1(0) = \mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle, \quad 2\sigma = \sqrt{2N_0[\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle]},$$

and

$$\mathsf{SNR} = \frac{\hat{s}_0(0) - \hat{s}_1(0)}{2\sigma} = \frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle}{\sqrt{2N_0[\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle]}} = \sqrt{\frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle}{2N_0}} = \mathsf{SNR}^*.$$

Note also that the minimax threshold is

$$\theta = \frac{\hat{s}_0(0) + \hat{s}_1(0)}{2} = \frac{\mathcal{E}_0 - \mathcal{E}_1}{2}. \tag{4.6}$$

[2] In fact, the receiver described above is optimum not just among all possible receivers that use linear filters, but among all possible receivers, whether with linear filters or nonlinear filters, or any other kind of processing of the sample value(s), i.e. with structures different from the one described in the figure above.

Thus, for *equal-energy* signals, including as a special case, *antipodal* signals $s_1(t) = -s_0(t)$, the threshold θ is 0. More generally, we have that

$$\hat{s}_0(\tau) = \int_{-\infty}^{\infty} h_m(\tau - t)s_0(t)\,dt = \int_{-\infty}^{\infty} [s_0(t-\tau) - s_1(t-\tau)]s_0(t)\,dt = R_{s_0}(\tau) - R_{s_0,s_1}(\tau) \qquad (4.7)$$

$$\hat{s}_1(\tau) = \int_{-\infty}^{\infty} h_m(\tau - t)s_1(t)\,dt = \int_{-\infty}^{\infty} [s_0(t-\tau) - s_1(t-\tau)]s_1(t)\,dt = R_{s_1,s_0}(\tau) - R_{s_1}(\tau) \qquad (4.8)$$

where $R_x(\tau) = \int_{-\infty}^{\infty} x(t+\tau)x(t)\,dt = \int_{-\infty}^{\infty} x(t)x(t-\tau)\,dt$ denotes the *autocorrelation* function of signal $x(t)$, and $R_{x,y}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)y(t)\,dt = \int_{-\infty}^{\infty} x(t)y(t-\tau)\,dt$ denotes the *cross-correlation* function of signals $x(t)$ and $y(t)$. Note that $R_{x,y}(\tau) = R_{y,x}(-\tau)$ and that $R_{x,y}(0) = R_{y,x}(0) = \langle x, y\rangle$. Thus, the respective responses $\hat{s}_0(t)$ and $\hat{s}_1(t)$ of the matched filter to the received signals $s_0(t)$ and $s_1(t)$ can be expressed in terms of the correlation functions of the signals $s_0(t)$ and $s_1(t)$. In fact, combining (4.7) and (4.8) and doing a little algebra shows that the response of the canonical matched filter to the *difference signal* $s(t) = s_0(t) - s_1(t)$ is the *signal separation* $\hat{s}(t) = \hat{s}_0(t) - \hat{s}_1(t) = R_s(t)$. Thus, the maximum signal separation at the sampling time 0 that the canonical matched filter is providing can be interpreted as a natural consequence of the fact that the *signal separation* $\hat{s}(t)$ at the filter output is an autocorrelation function; and autocorrelation functions have a peak at the origin. Note that all of the above results are also applicable to more general matched filters with impulse response $h(t) = \lambda[s_0(T_0 - t) - s_1(T_0 - t)]$, $\lambda > 0$ and sampling time T_0 with the following obvious changes.

$$\hat{s}_0(T_0) = \lambda[\mathcal{E}_0 - \langle s_0, s_1\rangle], \quad \hat{s}_1(T_0) = \lambda[\langle s_0, s_1\rangle - \mathcal{E}_1], \quad \sigma^2 = \frac{\lambda^2 N_0}{2}\left[\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle\right], \quad \theta = \lambda\frac{\mathcal{E}_0 - \mathcal{E}_1}{2}.$$

Notice that these changes do not affect SNR which still has value SNR^* as given by the right side of (4.1). Similarly, (4.7), and (4.8) become

$$\hat{s}_0(\tau) = \int_{-\infty}^{\infty} \lambda[s_0(t + T_0 - \tau) - s_1(t + T_0 - \tau)]s_0(t)\,dt = \lambda[R_{s_0}(\tau - T_0) - R_{s_0,s_1}(\tau - T_0)]$$

$$\hat{s}_1(\tau) = \int_{-\infty}^{\infty} \lambda[s_0(t + T_0 - \tau) - s_1(t + T_0 - \tau)]s_1(t)\,dt = \lambda[R_{s_1,s_0}(\tau - T_0) - R_{s_1}(\tau - T_0)]$$

that is, the filter outputs are delayed by T_0 (thus giving a maximum signal separation at T_0 instead of at 0) and multiplied by the positive gain factor λ.

4.3 The matched filter in the frequency domain

The transfer function of the canonical matched filter whose impulse response is $h_m(t) = s_0(-t) - s_1(-t)$ is $H_m(f) = \mathcal{F}[s_0(-t) - s_1(-t)] = S_0^*(f) - S_1^*(f) = S^*(f)$ where $S_i(f) = \mathcal{F}[s_0(t)]$ is the Fourier transform of $s_i(t)$ and $S(f) = \mathcal{F}[s(t)]$ is the Fourier transform of the *difference signal* $s(t)$.[3] Now, the the response of the matched filter to the $s(t)$ is the *signal separation* $\hat{s}(t) = \hat{s}_0(t) - \hat{s}_1(t)$ which can be described in the frequency domain as

$$\hat{S}(f) = \mathcal{F}[\hat{s}(t)] = \mathcal{F}[s(t) \star h_m(t)] = S(f)H_m(f) = S(f)S^*(f) = |S(f)|^2 \qquad (4.9)$$

which, as noted earlier, is the Fourier transform of the autocorrelation function $R_s(t)$ of the difference signal $s(t)$. The inverse Fourier transform or Fourier integral

$$\hat{s}_0(t) = \int_{-\infty}^{\infty} \hat{S}(f)\exp(j2\pi ft)\,df = \int_{-\infty}^{\infty} S(f)H_m(f)\exp(j2\pi ft)\,df = \int_{-\infty}^{\infty} S(f)S^*(f)\exp(j2\pi ft)\,df \qquad (4.10)$$

[3]The interested reader, if there is one, who has slogged through Appendix A of Lecture 3 and knows about power spectral densities should note that $S_i(f)$ is *not* the *power spectral density* or power spectrum of $s_i(t)$. The power spectrum of $s_i(t)$ is $|S_i(f)|^2 = \mathcal{F}[R_i(t)]$.

affords a different perspective on what the matched filter is doing and how it maximizes the signal separation at $t = 0$.

Digression: Consider two oscillators generating sinusoidal signals $A_0 \cos(2\pi f_0 t + \theta_0)$ and $A_1 \cos(2\pi f_1 t + \theta_1)$ whose sum has value $A_0 \cos(\theta_0) + A_1 \cos(\theta_1) \leq A_0 + A_1$ at $t = 0$. If we *delay* the signals from the oscillators by τ_0 and τ_1 respectively, then the sum of the delayed signals $A_0 \cos(2\pi f_0(t-\tau_0)+\theta_0)$ and $A_1 \cos(2\pi f_1(t-\tau_1)+\theta_1)$ has value $A_0 \cos(-2\pi f_0 \tau_0 + \theta_0) + A_1 \cos(-2\pi f_1 \tau_1 + \theta_1)$ at $t = 0$, and we can achieve maximum value $A_0 + A_1$ for the sum by setting the delays τ_i to have value $\theta_i/(2\pi f_i)$, $i = 0, 1$. This is can be thought of as *phase compensation*: the initial phases θ_0 and θ_1 of the oscillators are being compensated for (i.e. eliminated) by the delays that we have introduced so that the signal peaks align at $t = 0$. We apply this notion to the matched filter calculations. □

An integral is the limit of a sum, and thus the Fourier integral $x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t)\, df$ can be regarded as a prescription for generating $x(t)$ by summing the outputs of a large number of complex exponential oscillators $\exp(j2\pi f_i t)$ with amplitude $|X(f_i)|\Delta f_i$ and phase $\angle X(f_i)$. The limiting form of such a sum as the number of oscillators increases without bound and the amplitudes of the oscillators become infinitesimal is the Fourier integral. With this in mind, consider (4.10). One of the oscillators generating the input signal $s(t)$ is $\exp(j2\pi f_i t)$ with amplitude $|S(f_i)|\Delta f_i$ and phase $\angle S(f_i)$, i.e., $|S(f_i)|\Delta f_i \exp(j(2\pi f_i t + \angle S(f_i)))$. The matched filter has response $S^*(f_i) = |S(f_i)| \exp(-j\angle S(f_i))$ at frequency f_i. Thus, when the signal from this oscillator passes through the matched filter, the output signal $|S(f_i)|\Delta f_i \exp(j(2\pi f_i t + \angle S(f_i)))|S(f_i)| \exp(-j\angle S(f_i)) = |S(f_i)|^2 \Delta f_i \exp(j(2\pi f_i t)$ has been *phase compensated* perfectly and peaks at $t = 0$. In fact, since *all* the oscillators generating the input signal are phase compensated perfectly as they pass through the matched filter, all the complex exponentials comprising the signal response of the matched filter peak simultaneously at $t = 0$. But the matched filter does more: it changes each oscillator amplitude from $|S(f_i)|\Delta f_i$ at the input to $|S(f_i)|^2 \Delta f_i$. Now, for $x > 0$, $x^2 > x$ if $x > 1$ while $x^2 < x$ if $x < 1$. Thus, the matched filter can be viewed as *amplifying* those frequencies where the input signal is strong and *attenuating* those frequencies where the input signal is weak.[4] This is far better than amplifying all frequencies equally or, worse, having large gain in frequency bands where the signal is weak in a futile effort to bring up the signal strength. Recall that the noise is omnipresent in all frequency bands. Thus, no matter at what frequencies a filter has large gain, noise will be amplified and contribute to the noise variance. By having the large gains in frequency bands where the signal is strong, the matched filter creates a larger contribution to the signal strength at its output. In summary, the frequency-domain view of a matched filter is that of a system that phase compensates the input signal perfectly so that all frequency components of the difference signal are synchronized in time and peak simultaneously at $t = 0$, and a system that emphasizes the strong frequency components in the difference signal while de-emphasizing the weak frequency components.

4.4 Signal design

We have studied how a receiver can be designed to demodulate the given signals $s_0(t)$ and $s_1(t)$ optimally to achieve minimum error probability $P_e^* = Q(\sqrt{(\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1 \rangle)/2N_0})$, which depends only on the signal energies $\mathcal{E}_0$ and $\mathcal{E}_1$ and inner product $\langle s_0, s_1 \rangle$. If we can choose the signals, what should we choose them to be? Of course, increasing the signal energies is a good thing to do, but suppose that we have an *energy budget* of $\mathcal{E}_0 + \mathcal{E}_1 = 2\mathcal{E}_b$ where $\mathcal{E}_b$, the *energy per bit*, is fixed. An obvious choice is to set $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}_b$ and choose $s_0(t)$ and $s_1(t)$ to be *orthogonal* signals so that $\langle s_0, s_1 \rangle = 0$. This gives $P_e^* = Q(\sqrt{\mathcal{E}_b/N_0})$. An alternative set of orthogonal signals has $s_0(t)$ as a signal of energy $\mathcal{E}_0 = 2\mathcal{E}_b$ and $s_1(t) = 0$ which also achieves $P_e^* = Q(\sqrt{\mathcal{E}_b/N_0})$. A communication system using these signals is said to be using ON-OFF keying (OOK). But consider instead *antipodal* signals $s_1(t) = -s_0(t)$ of energy $\mathcal{E}_b$. Now, $\langle s_0, s_1 \rangle = -\mathcal{E}_b$ and hence $P_e^* = Q(\sqrt{2\mathcal{E}_b/N_0})$ which is far smaller than the $Q(\sqrt{\mathcal{E}_b/N_0})$ achieved by orthogonal signal sets. Antipodal signaling is often referred to as Binary Phase Shift Keying (BPSK) because sinusoidal signals $\pm\sqrt{2\mathcal{E}_b/T} \cos(2\pi f_c t)\mathrm{rect}(t/T)$ are commonly used, and changing from one signal to the other can be thought of as shifting the phase of the *carrier signal* at frequency f_c Hz by π or 180°. More precise nomenclature

[4] As an extreme case, the matched filter has zero response in frequency bands where the signal is not present. It thus completely suppresses the noise in those bands and prevents it from contributing to the noise variance at the filter output.

would be *antipodal* BPSK because non-antipodal forms of binary phase shift keying are also possible and used at times. For example, the signals $\sqrt{2\mathcal{E}_b/T}\cos(2\pi f_c t)\text{rect}(t/T)$ and $\sqrt{2\mathcal{E}_b/T}\sin(2\pi f_c t)\text{rect}(t/T)$ are *orthogonal* BPSK signals if $f_c T$ is an integer. The phase shifting is now by $\pm\pi/2$ or $\pm 90°$. More generally, BPSK signaling with phase shifts other than π is called *residual carrier* BPSK signaling since the signals can be decomposed into a unmodulated carrier signal (the residual carrier) and an antipodal BPSK signal that is orthogonal to the unmodulated carrier signal. The residual carrier makes it easier to set up *synchronization* systems, a vast topic that will have to remain unexplored in this course.

For a fixed energy budget of $\mathcal{E}_b$ per bit, antipodal signaling systems achieve the smallest possible error probability. But let us return to general signals $s_0(t)$ and $s_1(t)$ and express them as

$$\begin{aligned} s_0(t) &= \frac{s_0(t)+s_1(t)}{2} + \frac{s_0(t)-s_1(t)}{2} = s_c(t) + s_d(t) \\ s_1(t) &= \frac{s_0(t)+s_1(t)}{2} - \frac{s_0(t)-s_1(t)}{2} = s_c(t) - s_d(t) \end{aligned}$$

where $s_c(t)$ is the *common* signal that is *always* transmitted, and $\{s_d, -s_d\}$ is an antipodal signal set that constitutes the *data-dependent* portion of the transmitted signal. The common signal has energy $\frac{1}{4}(\mathcal{E}_0 + \mathcal{E}_1 + 2\langle s_0, s_1\rangle)$ while the data-dependent signal is $s_d(t) = \frac{1}{2}s(t)$ with energy $\frac{1}{4}(\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle)$. With canonical matched filter $h_m(t) = s(-t) = [s_0(-t) - s_1(-t)]$, the data-dependent signals $\pm s_d(t)$ produce contributions to the sample outputs of $\pm s_d \star h_m\big|_{t=0} = \pm\frac{1}{2}\langle s_0 - s_1, s_0 - s_1\rangle = \pm\frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle}{2}$ while the common signal contributes $s_c \star h_m\big|_{t=0} = \frac{1}{2}\langle s_0 + s_1, s_0 - s_1\rangle = \frac{\mathcal{E}_0 - \mathcal{E}_1}{2}$, that is, the minimax threshold! Put another way, we can write

$$\begin{aligned} \hat{s}_0(0) &= s_c \star h_m\big|_{t=0} + s_d \star h_m\big|_{t=0} = \frac{\mathcal{E}_0 - \mathcal{E}_1}{2} + \frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle}{2} = \mathcal{E}_0 - \langle s_0, s_1\rangle, \\ \hat{s}_1(0) &= s_c \star h_m\big|_{t=0} - s_d \star h_m\big|_{t=0} = \frac{\mathcal{E}_0 - \mathcal{E}_1}{2} - \frac{\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle}{2} = \langle s_0, s_1\rangle - \mathcal{E}_1 \end{aligned}$$

exactly as in (4.3) and (4.4). If we have two antipodal signals $\{s_d, -s_d\}$ of energy $\mathcal{E} = \frac{1}{4}(\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle)$, we can achieve minimum error probability of $Q(\sqrt{2\mathcal{E}/N_0}) = Q(\sqrt{(\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle)/2N_0})$ if we use a matched filter for them. But the canonical matched filter for signal set $\{s_d, -s_d\}$ is $s_d(-t) - (-s_d(-t)) = 2s_d(-t) = s(-t) = h_m(t)$. In other words, the canonical matched filter for signal set $\{s_d, -s_d\}$ is the same as the canonical matched filter for signal set $\{s_0, s_1\}$, and the minimum error probability achieved by signals $s_0(t)$ and $s_1(t)$ can be viewed as being entirely due to antipodal signals $\{s_d, -s_d\}$, the data-dependent part of the signals $s_0(t)$ and $s_1(t)$, while the energy in the common part $s_c(t)$ of the signals is merely "wasted" in producing the minimax threshold θ at the output of the matched filter. If we can somehow eliminate the common signal, we can achieve the same error probability by using the antipodal data-dependent signals only with much lower expenditure of energy per bit. But such a stratagem can be used only for channels that can transmit both positive and negative signals since what is left after eliminating the common signal is the antipodal data-dependent signal. Optical and infrared channels are examples of such unipolar channels since colloquially we can transmit light and heat but not dark and cold.

4.5 Summary

The optimum (minimum error probability) receiver for demodulating two signals $s_0(t)$ and $s_1(t)$ in additive white Gaussian noise is a matched filter (linear time-invariant system) followed by a sampler and a comparator that compares the sample value to a threshold. The matched filter impulse response is $\lambda[s_0(T_0 - t) - s_1(T_0 - t)]$ where $\lambda > 0$ is a gain constant chosen by the designer, and T_0 is the sampling time chosen by the designer. The *canonical* matched filter has $\lambda = 1$, $T_0 = 0$ and is generally a noncausal system. The error probability achieved by matched filtering is $P_e^* = Q(\sqrt{(\mathcal{E}_0 + \mathcal{E}_1 - 2\langle s_0, s_1\rangle)/2N_0})$. If the received energy is fixed at $\mathcal{E}_b$ per bit, then orthogonal signals achieve error probability $Q(\sqrt{\mathcal{E}_b/N_0})$ with matched filtering, while antipodal signals achieve even smaller error probability $Q(\sqrt{2\mathcal{E}_b/N_0})$ with matched filtering.