Appendix E

Substitution, Reduction, and Miller's Theorems

This appendix discusses three additional theorems that are especially useful in the simplification of circuits containing dependent sources. In our previous studies* we discussed the superposition principle and Thevenin's and Norton's theorems.

E.1 The Substitution Theorem

The *substitution theorem* states that if the voltage across a branch with nodes x and y of a network is v_{xy} and the current through this branch is i_{xy}, a different branch may be substituted in its place in the network provided that the voltage across the substitute branch is also v_{xy} and the current through it is also i_{xy}. The most common use of this theorem is to replace an impedance by a voltage or current source, or vice versa. The substitution theorem can best illustrated with the simple circuit and the substitute branches shown in Figure E.1.

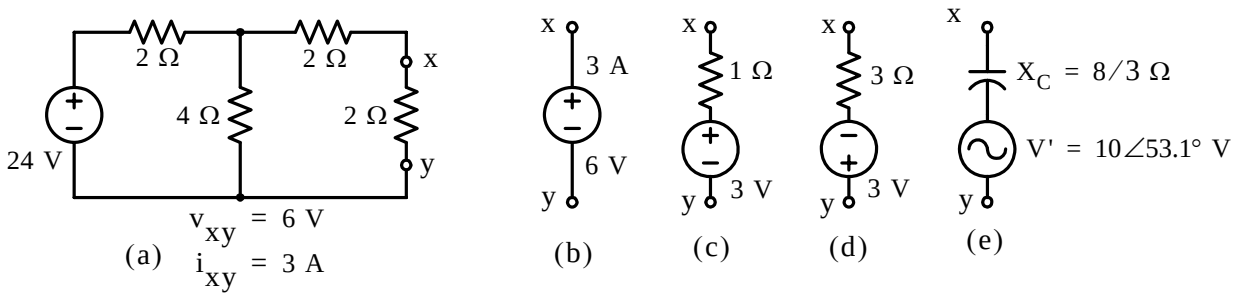

Figure E.1. Illustration of the substitution theorem

For the simple resistor circuit of Figure E.1(a) we find by series–parallel resistance combinations that $v_{xy} = 6$ V and $i_{xy} = 3$ A. According to the substitution theorem, the 2 Ω resistor across terminals x and y can be replaced with a source with a 6 V source as shown in Figure E.1(b) and the rest of the network will be unaffected. The current in the branch will be 3 A as before.

Other substitutions are possible also. For instance, the substitute branch may consist of a resistance of 1 Ω and a voltage source of 3 V as shown in Figure E.1(c), or it may consist of a resistance of 3 Ω and a voltage source of 3 V of opposite polarity as shown in Figure E.1(d).

* *For a detailed discussion of Thevenin's and Norton's theorems and the superposition principle please refer to Circuit Analysis I with MATLAB Applications, ISBN 978–0–9709511–2–0.*

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The voltage source in Figure E.1(a) could be DC or AC. If it is an AC source, we can substitute the $2\ \Omega$ resistor across terminals x and y with a branch that has capacitive reactance in series with a source V' as shown in Figure E.1(e). For this branch,

$$V_{xy} = -jX_C I_{xy} + V' \tag{E.1}$$

With the value $X_C = 8/3\ \Omega$, the value of the AC voltage source V' necessary to satisfy the substitution theorem must be such $V_{xy} = 6$ V and $I_{xy} = 3$ A and by substitution of these values into relation (E.1) we obtain

$$6 = -j(8/3)\,3 + V'$$

from which

$$V' = 6 + j8 = 10\angle 53.1°$$

as shown in Figure E.1(e).

Proof of the substitution theorem is based on the branch equations of the network. Operation of the whole network of B branches is defined by 2B simultaneous equations. Consider, for example, the bridge network shown in Figure E.2.

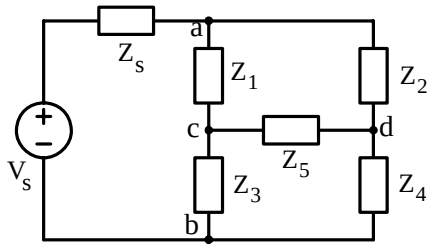

Figure E.2. Network for proof of the substitution theorem

The network of Figure E.2 has six branches and by application of Ohm's law,

$$\begin{aligned} V_{ab} &= Z_{ab}I_{ab} + V_s & V_{ac} &= Z_{ac}I_{ac} & V_{ad} &= Z_{ad}I_{ad} \\ V_{cb} &= Z_{cb}I_{cb} & V_{cd} &= Z_{cd}I_{cd} & V_{db} &= Z_{db}I_{db} \end{aligned} \tag{E.2}$$

If we assume that the source voltage V_S and six impedances are known constants, it becomes apparent that we have 6 equations with 12 unknowns, and thus we need 6 more equations to solve for all voltages and currents in each branch. We can obtain 3 additional equations by application of KCL, and the last three from KVL. These are shown below.

$$\begin{aligned} I_{ab} + I_{ac} + I_{ad} &= 0 & I_{ca} + I_{cd} + I_{cb} &= 0 & I_{da} + I_{dc} + I_{db} &= 0 \\ V_{ab} + V_{ac} + V_{cb} &= 0 & V_{ac} + V_{cd} + V_{da} &= 0 & V_{cb} + V_{bd} + V_{dc} &= 0 \end{aligned} \tag{E.3}$$

The 12 equations of (E.2) and (E.3) can now be solved simultaneously for the 12 unknowns.

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From the foregoing discussion we conclude that it is possible to write as many equations as there are unknowns. Thus, if the number of branches is B, the number of unknowns is 2B. Let us now suppose that the branch for which substitution is to be made contains an impedance and a source and the equation for this branch is

$$V_{ab} = Z_{ab}I_{ab} + V_S \tag{E.4}$$

that is, the voltage across this branch is V_{ab} and the current is I_{ab}. Other branch voltages in the network are V_{bc}, V_{cd}, and so on, and currents are I_{bc}, I_{cd}, and so on, and all these voltages and currents simultaneously satisfy the 2B equations of the network, including equation (E.4).

Next, let us assume a substitute branch that has some different value of impedance Z'_{ab}, and some different voltage source V'_S, but with Z'_{ab} and V'_S so related that the same values of V_{ab} and I_{ab} that satisfied equation (E.4) will also satisfy the new branch equation

$$V_{ab} = Z'_{ab}I_{ab} + V'_S \tag{E.5}$$

When this substitution is made, operation of the altered network will be defined by 2B equations, as before; (2B – 1) of the equations will remain the same but equation (E.5) will have replaced equation (E.4). However, since equation (E.5) is satisfied by the same values of voltage and current that satisfied equation (E.4), it follows that the whole group of simultaneous equations will be satisfied by the same voltages and currents as before. Voltages and currents of all branches of the network will therefore be the same as before substitution, which was to be proved.

The substitution theorem is especially useful in the simplification of circuits containing dependent sources. Consider for example the current–controlled voltage source and the voltage–controlled current source shown in Figure E.3.

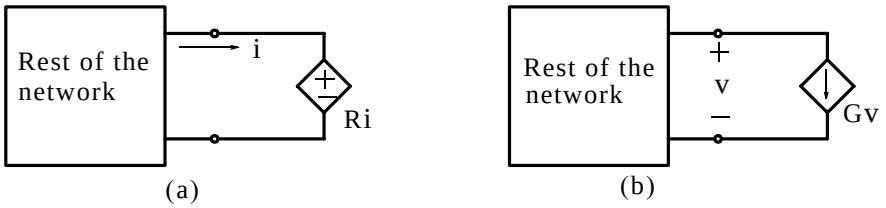

Figure E.3. Application of the substitution theorem in current– and voltage–controlled dependent sources.

The network of Figure E.3(a) can be any network having the terminal current i, and Ri is the voltage of a current–controlled voltage source. Figure E.3(b) shows the conditions under which the dual form can be applied. In this case the voltage–controlled current source Gv, which is connected across the voltage v. As another example, let us consider the equivalent circuit of some amplifier shown in Figure E.4(a).

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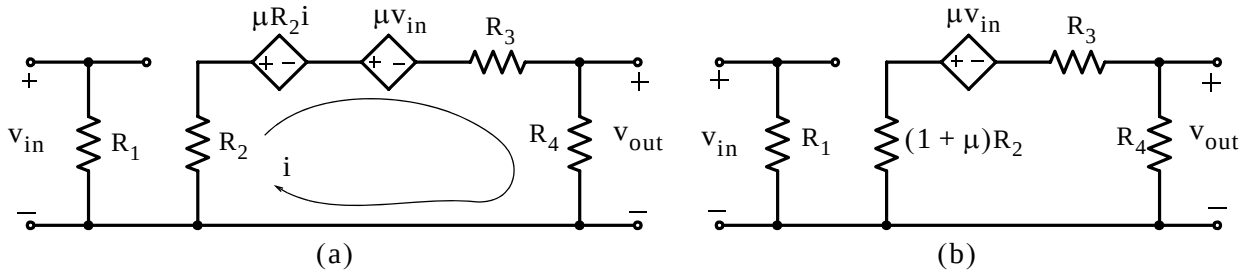

Figure E.4. Application of the substitution theorem in an amplifier equivalent circuit.

In Figure E.4(a), the currents and voltages are not changed if the dependent source $\mu R_2 i$ is replaced by a resistance μR_2, and the simplified equivalent circuit is as shown in Figure E.4(b).

A special case of the substitution theorem is the *source absorption theorem** and has two dual forms, the *voltage source absorption* and the *current source absorption* theorems.

The voltage source absorption theorem is illustrated in Figure E.5.

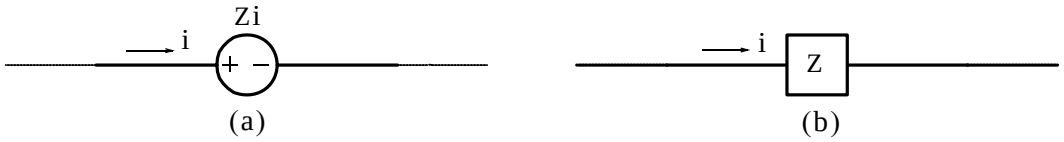

Figure E.5. Illustration of the voltage source absorption theorem

As shown in Figure E.5, the voltage source absorption states that if in a branch of the network in which the current through it is i, the voltage source Zi can be replaced by an impedance Z.

Example E.1

Use the voltage source absorption theorem to simplify the branch AB shown in Figure E.6.

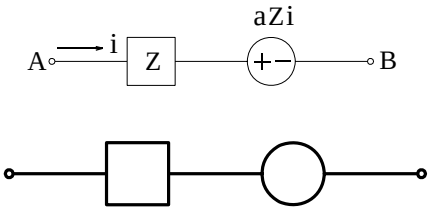

Figure E.6. Branch for Example E.1 to be simplified

* *This theorem is analogous to the absorption theorem in Boolean algebra which states that* $A = (\bar{A} + B) = AB$

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Solution:

Application of the voltage source absorption theorem yields the simplified branch shown in Figure E.7(c) below.

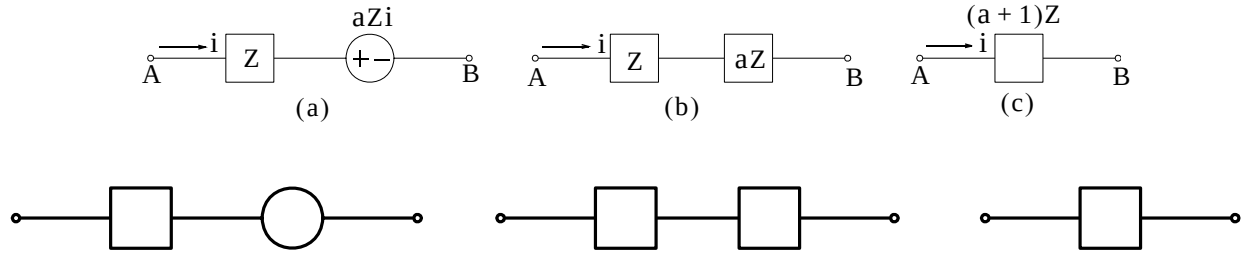

Figure E.7. Steps for simplification of the branch for Example E.1

The current source absorption theorem is illustrated in Figure E.8.

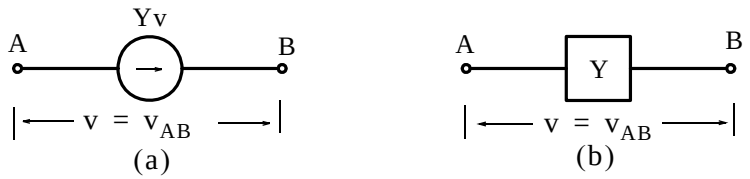

Figure E.8. Illustration of the current source absorption theorem

As shown in Figure E.8, the current source absorption states that if in a branch of the network in which the voltage across it is v, the current source Yv can be replaced by an admittance Y.

Example E.2

Use the current source absorption theorem to simplify the parallel combination shown in Figure E.9.

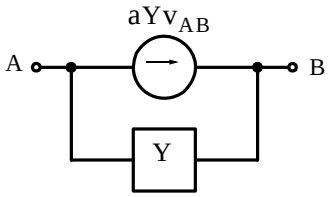

Figure E.9. Branch for Example E.2 to be simplified

Solution:

Application of the current source absorption theorem yields the simplified branch shown in Figure E.10(c) below.

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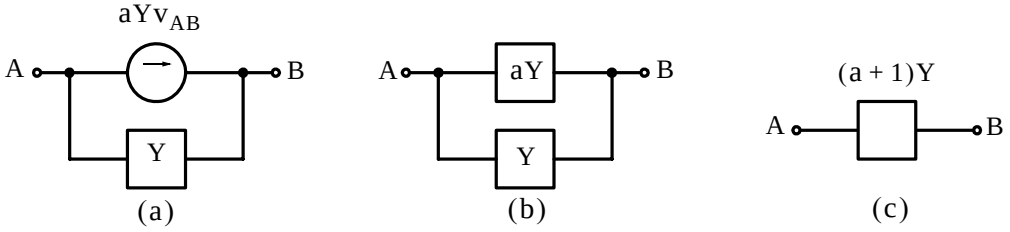

Figure E.10. Steps for simplification of the parallel combination of Example E.2

E.2 The Reduction Theorem

The *reduction theorem* is essentially an extension of the substitution theorem. This theorem applies to network configurations such as those shown in Figure E.11.

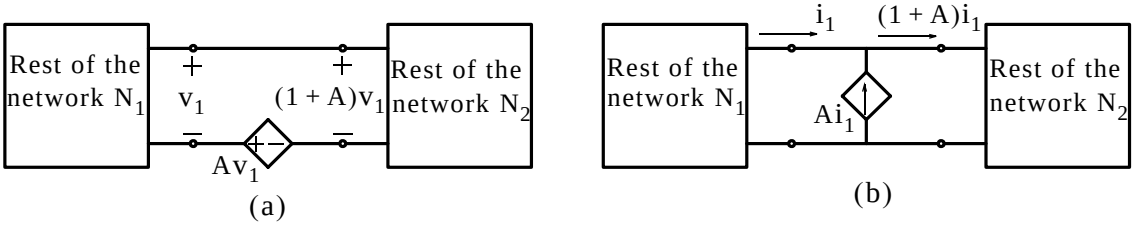

Figure E.11. Network configurations that can be simplified by the reduction theorem

The reduction theorem applies to the network configurations shown in Figure E.11. The network N_1 in Figure E.11(a) is any linear two–terminal network having the terminal voltage v_1, and N_2 is any other linear two–terminal network. The constant A of the voltage–controlled voltage source Av_1 is any real number, positive or negative. The two networks and the voltage source are connected in series so that the voltages v_1 and Av_1 are additive with respect to the loop that they form. The theorem states that all currents in N_1 and N_2 remain unchanged if the source Av_1 is replaced with a short circuit and if:

1. Each resistance, inductance, reciprocal capacitance, and voltage source in Network N_1 is multiplied by $1 + A$

   or

2. Each resistance, inductance, reciprocal capacitance, and voltage source in Network N_2 is divided by $1 + A$.

Proof of the reduction theorem is based on the form of the loop equations. Obviously, if all resistances, inductances, reciprocal capacitances, and voltage sources are multiplied by the same arbi-

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trary factor, then every term in the set of loop equations is multiplied by the same factor, and all the loop currents remain unchanged. Thus, if all the parameters and voltage sources in N_1 are multiplied by $1 + A$ as specified in Part 1 of the theorem, and if Av_1 is replaced with a short circuit, then the voltage applied to N_2 is still $(1 + A)v_1$, and all currents in N_1 and N_2 remain unchanged. All voltages in N_1 are multiplied by the factor $1 + A$, but the voltages in N_2 are not changed. Any current sources in N_1 and N_2 are left unchanged in the reduction process.

By the same reasoning process, after the transformation described above is completed, all resistances, inductances, reciprocal capacitances, and voltage sources in the composite network $N_1 - N_2$ can be divided by the factor $1 + A$ without changing any of the currents in $N_1 - N_2$. Thus, Part 2 of the theorem is proved.

When all currents and voltages in the network are sinusoidal and of the same frequency, the constant of the controlled source may be expressed as a complex number; the reduction theorem holds in this case also.

The dual form of the reduction theorem is applicable to the network configuration shown in Figure E.11(b). It states that all voltages in the network of Figure E.11(b) remain unchanged if the current source Ai_1 is replaced with an open circuit and if:

1. All conductances, capacitances, reciprocal inductances, and current sources in N_1 are multiplied by $1 + A$

   or

2. All conductances, capacitances, reciprocal inductances, and current sources in N_2 are divided by $1 + A$

Proof of the dual form of the theorem follows from nodal equations and is dual to the proof outlined above. In applying the reduction theorem it is essential that the two networks N_1 and N_2 be properly identified before application of this theorem. The two necessary requirements are:

a. The terminal voltage (or current) of the network identified as Network N_1 must be the controlling quantity for the source to be eliminated; and
b. No current may enter or leave either network through any terminal other than the two terminals by which the networks and controlled source are joined in series (or parallel). It should be noted that connections joining separate parts of a network carry no current and can therefore be ignored in respect to item b.

A useful application of the reduction theorem is provided by the *cascode amplifier* shown in Figure E.12.

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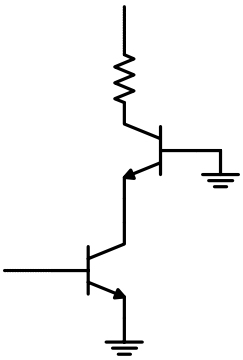

Figure E.12. Typical cascode amplifier with BJTs

Figure E.12 shows a partial circuit of a cascode amplifier where a common–base bipolar transistor sits on top of a common–emitter bipolar transistor. Of course, a cascode amplifier may use MOSFETs of CMOS devices. To illustrate an application of the reduction theorem, let us consider the the incremental model of some cascode amplifier shown in Figure E.13.

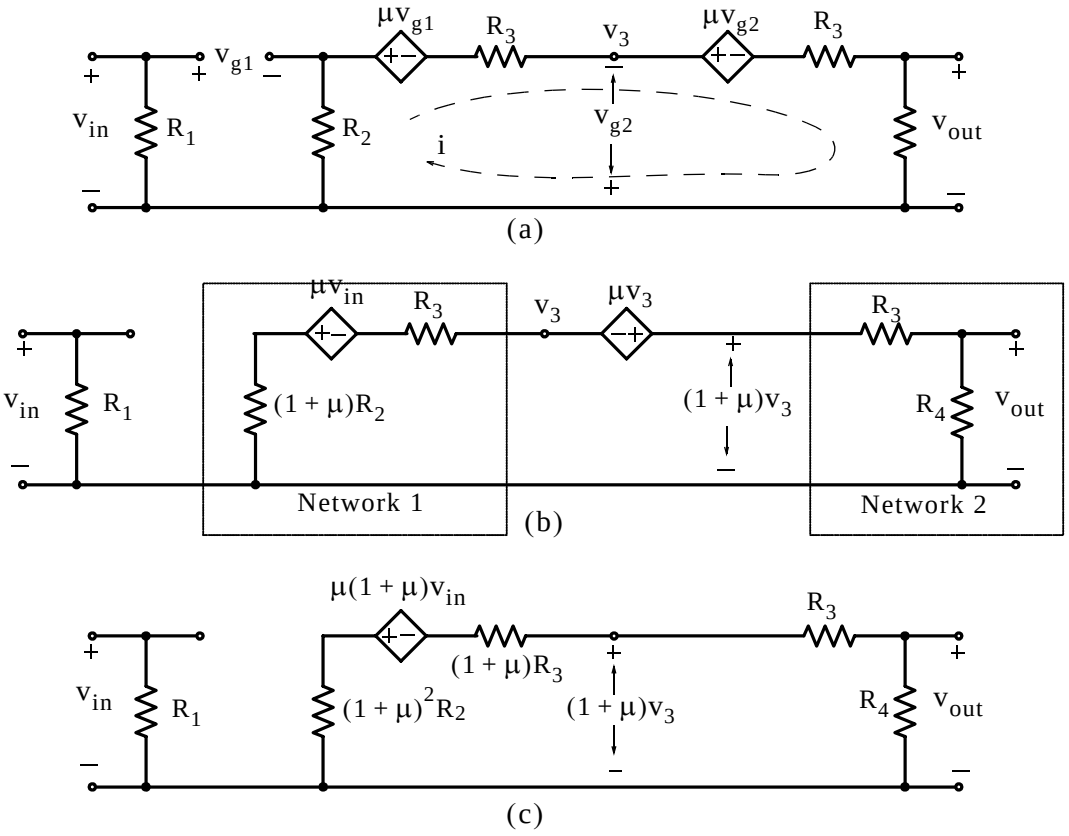

Figure E.13. Incremental model of a cascode amplifier and simplification by application of the reduction theorem

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From the circuit of Figure E.13(a) we observe that $v_{g1} = v_{in} + R_2 i$; hence the circuit can be simplified by applying the substitution theorem to obtain the equivalent circuit shown in Figure E.13(b). This circuit has been separated into two networks, in N_1 and N_2, connected in series with the controlled source μv_3, in preparation for the reduction theorem. All the conditions of the reduction theorem are satisfied by this circuit; hence applying Part 1 of the voltage–source form of the theorem yields the reduced network shown in Figure E.13(c).

The emitter follower shown in Figure E.14(a) provides another illustration of the use of the reduction theorem in circuit analysis.

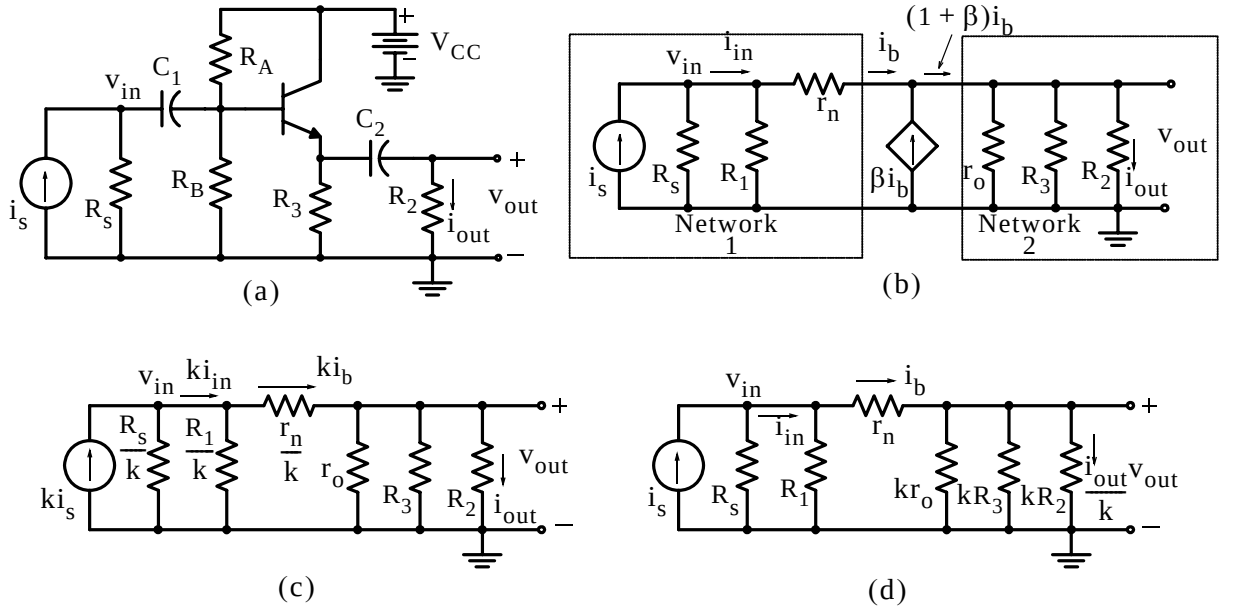

Figure E.14. Simplification of the emitter follower by application of the reduction theorem

As we know from Chapter 3, the principal properties of the emitter follower are that it provides a relatively low output resistance, it is capable of providing a relatively high input impedance, and it provides current and power amplification but no voltage amplification. Figure E.14(b) shows an incremental model for the circuit that is valid in the range of frequencies where the coupling capacitors act as short circuits and where the performance of the transistor is independent of frequency. The hybrid representation for the transistor is used with $\mu = 0$. The resistance R_1 represents the parallel combination of resistors R_A and R_B, and the circuit is separated into two networks in preparation for the reduction theorem.

Application of Part 1 of the current–source form of the theorem yields the reduced circuit shown in Figure E.14(c). For simplicity, the factor $1 + \beta$ is denoted as k in Figure E.14(c); in practice k is essentially equal β. Dividing the resistances R_S, R_1, and r_n, by k is equivalent to multiplying

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the corresponding conductances by this factor. Thus, application of the reduction theorem yields the simplified circuit of Figure E.14(c).

The current amplification of the circuit is accounted for by the fact that the input signal current is multiplied by the factor k in the reduced circuit. It is clear from Figure E.14(c) that the forward voltage transmittance of the emitter follower is less than unity, and for large values of β the output resistance of the emitter follower is very small.

If Part 2 of the theorem is applied instead of Part 1, the circuit of Figure E.14(d) is obtained; this circuit is convenient for studying relations at the input terminals since the input voltage and current are unmodified. For large values of β the output resistance of the emitter follower is very large.

E.3 Miller's Theorem

Miller's theorem states that under certain conditions, to be established below, the networks of Figures E.15(a) and E.15(b) are equivalent.

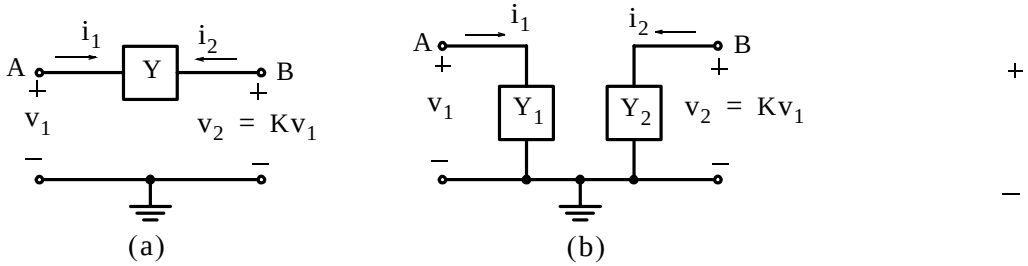

Figure E.15. Illustration for Miller's theorem

In the networks of Figure E.15, the constant K denotes the voltage gain from Node A to Node B, that is,

$$K = v_2/v_1 \tag{E.6}$$

or

$$v_2 = Kv_1 \tag{E.7}$$

At Node A of Figure E.15(a), we observe that

$$i_1 = Y(v_A - v_B) = Y(v_1 - v_2) = Y(v_1 - Kv_1) = Yv_1(1-K) \tag{E.8}$$

and at Node A of Figure E.15(b), we obtain,

$$i_1 = Y_1 v_1 \tag{E.9}$$

For equivalence, relations (E.8) and (E.9) must be equal. Thus,

$$Y_1 = Y(1-K) \tag{E.10}$$

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At Node B of Figure E.15(a), we observe that

$$i_2 = -i_1 = -Y(v_1 - v_2) = Y(v_2 - v_1) = Y(v_2 - v_2/K) = Yv_2(1 - 1/K) \tag{E.11}$$

and at Node B of Figure E.15(b), we obtain,

$$i_2 = Y_2 v_2 \tag{E.12}$$

For equivalence, relations (E.11) and (E.12) must be equal. Thus,

$$Y_2 = Y(1 - 1/K) \tag{E.13}$$

We should remember that for the equivalent circuit of Figure E.15(b) the value of the gain K, once established, it cannot be changed. Thus, while Miller's theorem can be applied to find the input impedance, it should not be used to determine the output impedance of an amplifier since when determining the output impedance the input signal voltage v_S is normally set to zero volts.

Miller's dual theorem states that under certain conditions, to be established below, the networks of Figures E.16(a) and E.16(b) are equivalent.

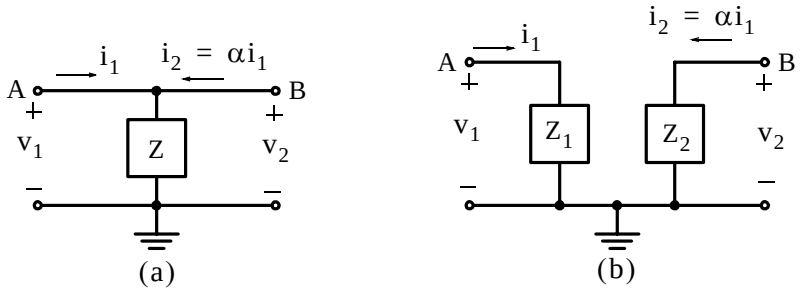

Figure E.16. Illustration for Miller's dual theorem

In the networks of Figure E.16, the constant α denotes the current gain from Node A to Node B, that is,

$$\alpha = i_2/i_1 \tag{E.14}$$

or

$$i_2 = \alpha i_1 \tag{E.15}$$

At Node A of Figure E.16(a), we observe that

$$v_1 = Z(i_1 + i_2) = Z(i_1 + \alpha i_1) = Zi_1(1 + \alpha) \tag{E.16}$$

and at Node A of Figure E.16(b), we obtain,

$$v_1 = Z_1 i_1 \tag{E.17}$$

For equivalence, relations (E.16) and (E.17) must be equal. Thus,

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$$Z_1 = Z(1+\alpha) \tag{E.18}$$

At Node B of Figure E.16(a), we observe that

$$v_2 = v_1 = Zi_1(1+\alpha) \tag{E.19}$$

and at Node B of Figure E.16(b), we obtain,

$$v_2 = Z_2 i_2 = Z_2 \alpha i_1 \tag{E.20}$$

For equivalence, relations (E.19) and (E.20) must be equal. Thus,

$$Z_2 = Z(1+1/\alpha) \tag{E.21}$$

Example E.3

For the circuit of Figure E.17, it is given that $R_1 = 100R_2$. Use Miller's theorem to replace this circuit with an equivalent circuit where resistor R_1 is being replaced with another resistor in parallel with resistor R_S. Make any reasonable assumptions.

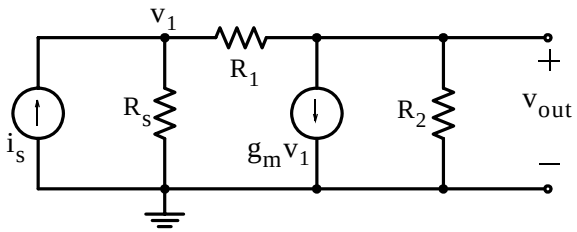

Figure E.17. Circuit for Example E.3

Solution:

Since $R_1 \gg R_2$ it is reasonable to assume that practically all the current of the current source flows through resistor R_2 and thus the output voltage is

$$v_{out} = -g_m v_1 R_2 = Kv_1$$

from which

$$K = -g_m R_2$$

Figure E.18 shows the equivalent circuit after application of Miller's theorem where using relation (E.10) we obtain

$$R'_1 = R_1/(1-K) = R_1/(1+g_m R_2)$$

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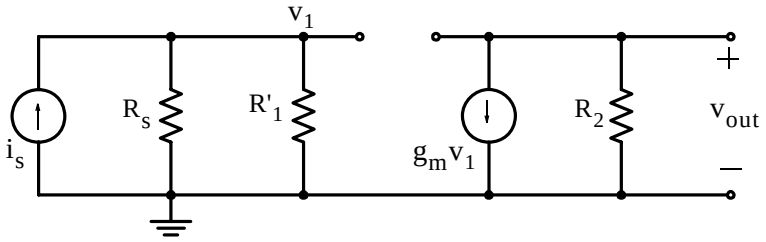

Figure E.18. Miller equivalent circuit for the circuit of Figure E.17

Being familiar with Miller's theorem helps us in better understanding the so–called *Miller effect*. This effect is very pronounced in amplifier circuits where the input capacitance caused by negative feedback from output to the input causes a significant increase in capacitance. As an illustration, let us consider the amplifier of the circuit in Figure E.19.

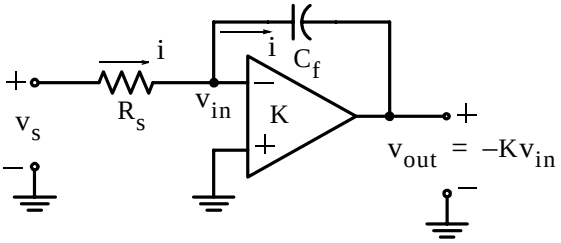

Figure E.19. Amplifier to illustrate the Miller effect

Assuming a sinusoidal input, the current i is

$$i = \frac{v_{in} - v_{out}}{1/j\omega C_f} = \frac{v_{in} + Kv_{in}}{1/j\omega C_f} = \frac{v_{in}(1+K)}{1/j\omega C_f}$$

and the input admittance is

$$Y_{in} = \frac{i}{v_{in}} = j\omega C_f(1+K) \qquad \text{(E.22)}$$

Relation (E.22) indicates that the input voltage v_{in} sees an increase in capacitance by a factor of $C_f(1+K)$.

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