Lecture 1 : Signal Representation and Sampling

Learning Objectives:

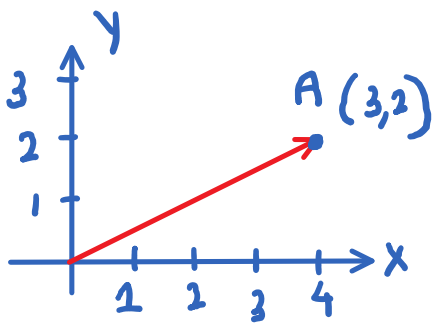
1. **Compute DTFT, DFT of sequences.**
2. **Verify the parameters of window function through simulations.**
3. **Simulate and verify the implications of spectral sampling.**

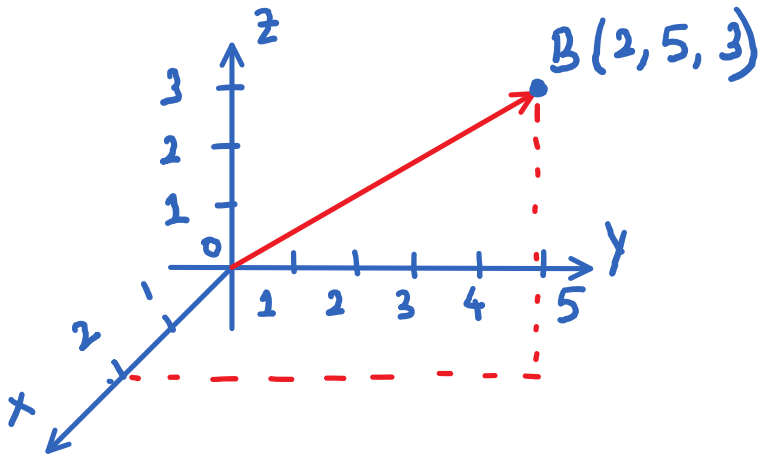
1. Signal Representation

This section is a brief review of concepts we learned in ECE 210 and ECE 310. We will look at Fourier Transform of signals. We will review DTFT and DFT concepts. Signal representation in the Fourier (frequency) domain offers us insights useful for analysis in signal processing applications and in the context of circuits can provide a simple means to test and verify circuit operation.
IN order to look at how sinusoidal and comlex exponential functions are building block of signals, let us look at the following examples

1.1 "Signal" Representation in space:

Consider two points A and B in 2-D and 3-D space respectively as shown in Fig. 1.1 below. One can visualize the point in 3-D space as a being a projector in a ECEB classroom. The location of the points is specified in terms of the unit vectors along each axis.





(a) $\mathbf{x_a} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$

(b) $\mathbf{x_b} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$

Fig. 1.1 A point in 2-D and 3-D space.

Generalizing to $n-D$ we have, $\mathbf{x} = \sum_1^n a_k \mathbf{u_k}$ **(1)**

We can make the following observations:

- a_k assigns a weight to the direction vector $\mathbf{u_k}$
- The summation in (1) gives an idea of the contribution of each direction to the final location.
- If $a_k = 0$, then the corresponding direction is not "present" in the final location. For example, in 2-D, $\mathbf{x_a} = \begin{bmatrix}2\\0\end{bmatrix}$ does not contain the $y-$direction, $\begin{bmatrix}0\\1\end{bmatrix}$.

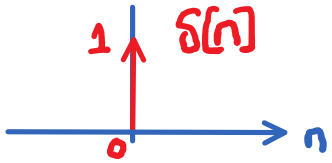
This example illustrated how we can combine "simple" directions to obtain the location of a point. Let us look at some examples from ECE 310.

1.2 Signal representation using delta functions

In ECE 310 we looked at discrete time signals represent as a combination of delta functions. The Kronecker delta function is defined as,

- **Kronecker Delta Function:**

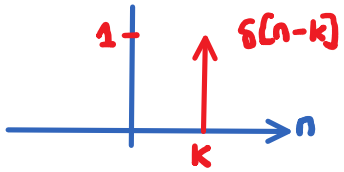
$$\delta[n] = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$$



Similarly a shifted delta function can be defined as,

- **Shifted delta function:**

$$\delta[n-k] = \begin{cases} 1, n = k \\ 0, n \neq k \end{cases}$$



The "simple" delta and shifted delta functions can be used to realize discrete time sequences as shown below,

Example 1.2.1: $x[n] = [1, 2, 3, 4, \dots]$

$\uparrow_{n=0}$

$$= \delta[n] + 2\delta[n-1] + 3\delta[n-2] + \cdots$$

$$x[n] = \sum_{k=1}^{\infty} k \times \delta[n-k]$$

Contribution of each delta function.

2. Fourier Series representation of periodic functions

Periodic signals of period $\mathbf{T_0}$ can be synthesized from a combination of periodic signals with same period. Let $\mathbf{x_1(t)}$ and $\mathbf{x_2(t)}$ be two signals of period $\mathbf{T_0}$ i.e.,

$$\mathbf{x_1(t) = x_1(t + T_0),\ x_2(t) = x_2(t + T_0)}$$

Consider a signal $\mathbf{x(t)}$ given by,

$$\mathbf{x(t) = x_1(t) + x_2(t)}$$

we then have,

$$\mathbf{x(t + T) = x_1(t + T_0) + x_2(t + T_0)}$$

$$\mathbf{\Rightarrow x(t + T_0) = x(t).}$$

Hence $x(t)$ is periodic with period $\mathbf{T}$.

Example 1.3.1: Consider the signal given by,

$$\mathbf{x(t) = \sum_{k=1}^{N} \frac{1}{2k} \sin\left(\frac{2\pi k}{T_0} t\right).} \qquad (1)$$

The signal $x(t)$ is a sum of sinusoidal functions. Let the fundamental period $\mathbf{T_0 = 1.}$ Fig. 1.2 shows the signal for four values of $N = 1,2,4,100$. From Fig. 1.2 as N increases we approach the waveform shown in Fig. 1.2 (d). In (1) the coefficients $\frac{1}{2k}$ give us an idea of the "contribution" of each **harmonically related** sinusoid.

2.1 From sinusoids to exponentials

Consider the general case of a periodic signal expressed as sum of sines and cosines. We then have,

$$\mathbf{x(t) = C_0 + \sum_{k=0}^{N} C_k \cos\left(\frac{2\pi k}{T_0} t + \theta_k\right)}$$

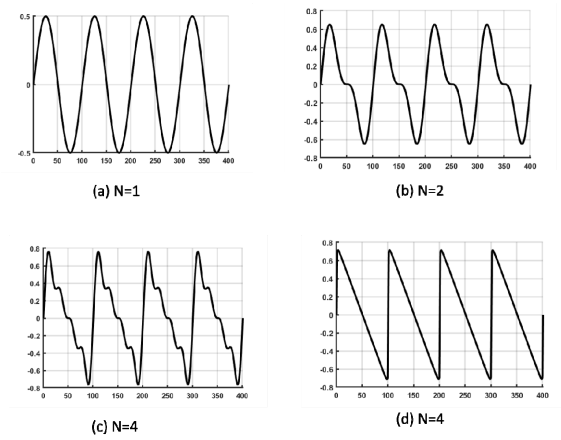


Fig. 1.2 Illustration of Fourier Series

The summation can rewritten as shown below,

$$x(t) = C_0 + \sum_{k=1}^{N} C_k e^{\frac{j2\pi k}{T_0}t + j\theta_k} + C_k e^{-\frac{j2\pi k}{T_0} - j\theta_k}$$

$$x(t) = \sum_{k=-N}^{N} X[k] e^{\frac{j2\pi k}{T_0}t} = \sum_{k=-N}^{N} X[k] e^{jk\omega_0 t}$$

Where,

$$X[k] = \begin{cases} C_0, & k = 0 \\ C_k e^{j\theta_k}, & k > 0 \\ C_k e^{j\theta_k}, & k < 0 \end{cases}$$

2.2 Continuous Time Fourier Series (CTFS)

Generalizing we have the Continuous Time Fourier Series **(CTFS)** synthesis and analysis equations,

CTFS Synthesis Equation: $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$

CTFS Analysis Equation: $X[k] = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t}\, dt$

Note:

- $x(t)$ is synthesized from harmonically related exponentials $x(t)$e^(−jk
- Expressing in terms of exponentials allows $x(t)$ to be complex valued.
- Allows the summation to be computed as a sum of geometric series!

3. Continuous Time Fourier Transform (CTFT)

CTFS provides a means to represent periodic signals in terms of complex exponentials. This idea can be extended to continuous time aperiodic signals by taking the period $T_0 \to \infty$. In this case the harmonically related frequencies are replaced by a continuum of frequencies. Hence the summation in CTFS synthesis equation takes the form of an integral and the limits on the integral in CTFS amalysis equation change to $\pm\infty$.

CTFT Synthesis Equation: $\mathbf{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{j\omega t}d\omega}$

CTFT Analysis Equation: $\mathbf{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt}$

3. Discrete Time Fourier Series (DTFS)

The idea of expressing a continuous-time signal $\mathbf{x(t)}$ as a combination of exponential signals can hbe extended to discrete-time (DT) sequence. Consider a periodic DT sequence with fundamental period **N**,

$\mathbf{x(n) = x(n + N)}$ where, n and N are integers.

The DTFS of $\mathbf{x[n]}$ is then given by,

DTFS Synthesis Equation: $\mathbf{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{\frac{j2\pi k}{N}n}}$

DTFS Analysis Equation: $\mathbf{X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{j2\pi k}{N}n}}$.

In the above expressions, **k**, **N** are integers**.** As in the case of CTFS, the DTFS expresses a DT signal as a combination of **harmonically related** DT exponentials.

Note:

- $\frac{\omega_k}{\pi} = \frac{2k}{N}$ must be a rational quantity.
- $\mathbf{X[k]}$ **i**s periodic with period **N**.
- Also note that $\boldsymbol{X[k + 2\pi] = X[k]}$.

4. Discrete Time Fourier Transform (DTFT)

A discrete time aperiodic signal $\mathbf{x[n]}$ **c**an be seen as a periodic signal with period $\mathbf{N \to \infty}$**.** In this case the frequency variable **ω** takes a continuum of values. Hence the summation in the DTFS synthesis equation is replaced by an integral and the DTFS analysis equation contains an infinite sum.

DTFT Synthesis Equation: $\mathbf{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega t}d\omega}$

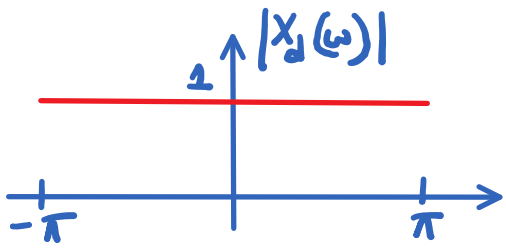
DTFT Analysis Equation: $\mathbf{X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-\omega n}}$.

Note:

- $\mathbf{X_d(\omega)}$ is periodic with period $\mathbf{2\pi}$**.**
-
- $\mathbf{X_d(\omega)}$ is a continuous function of **ω**.

Example 4.1: Compute the DTFT of $x[n] = \delta[n]$.

$$\mathbf{X_d(\omega)} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$



Example 4.2: Compute the DTFT of $x[n] = e^{j\omega_0 n}$.

Recall from ECE 210:

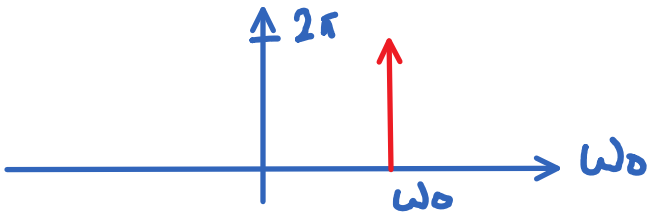
$$\int_{-\infty}^{\infty} \delta(t-a) f(t) = f(a) \qquad (1)$$

Also the DTFT synthesis (inverse DTFT) equation is,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{X_d(\omega)}\, e^{j\omega n} d\omega \qquad (2)$$

From (1) and (2),

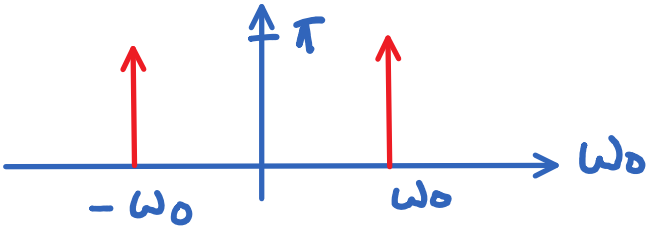
$$X_d(\omega) = 2\pi\delta(\omega - \omega_0)$$



Example 4.3: Compute the DTFT of $x[n] = \cos(\omega_0 n)$.

$$\cos(\omega_0 n) = \frac{(e^{j\omega_0 n} + e^{-j\omega_0 n})}{2}.$$

$$\Rightarrow X_d(\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



Example 4.4: Compute the DTFT of $x[n] = [u[n] - u[n-N+1]$

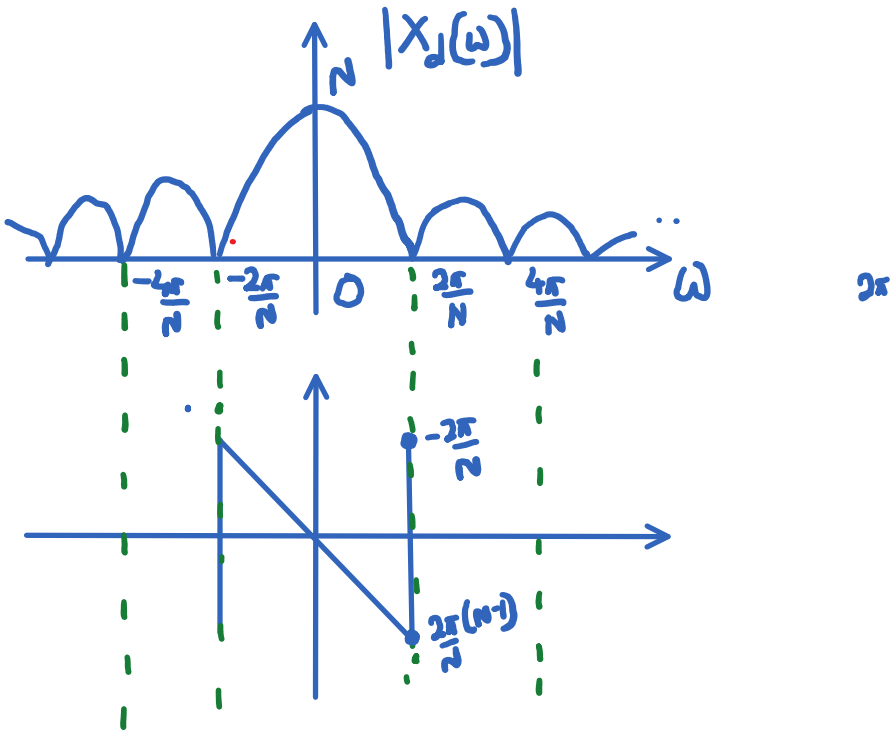
$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X_d(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega nN}}{1 - e^{-j\omega n}}$$

$$= \frac{e^{-\frac{j\omega nN}{2}} \sin\left(\frac{N\omega}{2}\right)}{e^{-\frac{j\omega n}{2}} \sin\left(\frac{\omega}{2}\right)}$$

$$X_d(\omega) = \frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-\frac{j\omega(N-1)}{2}}$$



5. Discrete Fourier Transform (DFT)

As discussed earlier the DTFT of a signal is given by,

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

The following are the issues with DTFT:

- The summation is of infinite length.
- Frequency variable ω is continuous.

Solution: Take N samples of $X_d(\omega)$ in an interval of length 2π.

The DFT can be seen as sampled DTFT and is given by,

$$\mathbf{X[k] = X_d\left(\omega = \frac{2\pi k}{N}\right)}$$

$$\boxed{\mathbf{X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi k}{N} n}, k = 0, 1, \dots, N-1}}$$

Hence the DFT representation of a finite length signal is given by,

$$\boxed{\begin{aligned} &\textbf{DFT :} && \mathbf{X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi k}{N} n}}, && \mathbf{k = 0, 1, \dots, N-1} \\ &\textbf{Inverse DFT :} && \mathbf{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi k}{N} n}}, && \mathbf{k = 0, 1, \dots, N-1} \end{aligned}}$$

Note:

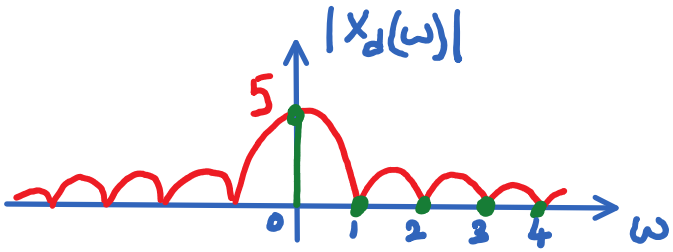
- Compare the DFT representation with DTFS. By computing DFT we implicitly assume that the length **N** signal is periodic with period **N.**
- The DFT computation can introduce "non-idealities"/artefacts that may have to be taken care of whille performing frequency analysis

Example 5.1: Compute the DTFT, $X_d(\omega)$ of $x[n] = u[n] - u[n-5]$. Plot $|X_d(\omega)|$ and indicate the samples of $X[k]$, where $X[k]$ is the DFT of $x[n]$.

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{4} e^{-j\omega n}$$

$$X_d(\omega) = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}} = \frac{e^{\frac{-j5\omega}{2}}\left(e^{\frac{j5\omega}{2}} - e^{-\frac{j5\omega}{2}}\right)}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}$$

$$\boxed{X_d(\omega) = \frac{5 sinc\left(\frac{5\omega}{2}\right)}{sinc\left(\frac{\omega}{2}\right)} e^{-j2\omega}}$$



Hence, $\boxed{X[k] = [1,0,0,0,0].}$

What will happen of you compute a $N = 16$, length DFT of the given signal?

5. Effect of Windowing

Sinusoidal signals are often employed in testing of analog and mixed signal circuits. The signal that is used must of finite length since we cannot have an infinite length signal in practice. Consider the finite length signal shown below,

$$x[n] = \cos \omega_0 n, \qquad 0 \le n \le N-1$$

The signal $x[n]$ can be seen as a product of an infinite length cosine signal and a finite length window function as shown below,

$$x[n] = x_1[n] \times w[n],$$

$$\text{where, } x_1[n] = \cos \omega_0 n, \qquad -\infty < n < \backslash\text{infty}$$

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & else \end{cases}$$

The implications of windowing can be easily seen in the frequency. Since $x[n]$ is the product of two signals, the DTFT $X_d(\omega)$ of $x[n]$ is given by the following relationship,

$$X_d(\omega) = X_{1d}(\omega) * W_d(\omega)$$

$$\text{where, } * \rightarrow \text{convolution}$$

$$W_d(\omega), X_{1d}(\omega) \rightarrow \text{DTFT of } x[n], w[n]$$

The following figures illustrate the effect of applying a window on an infinite length cosine signal.

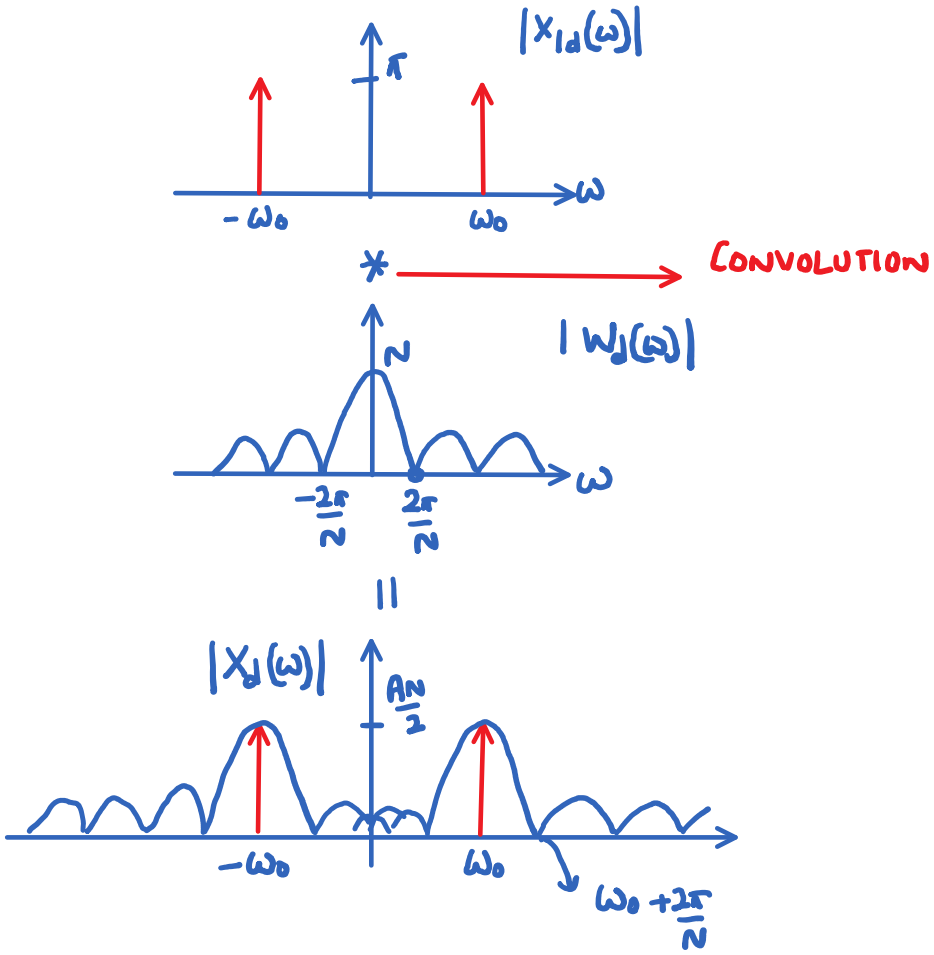
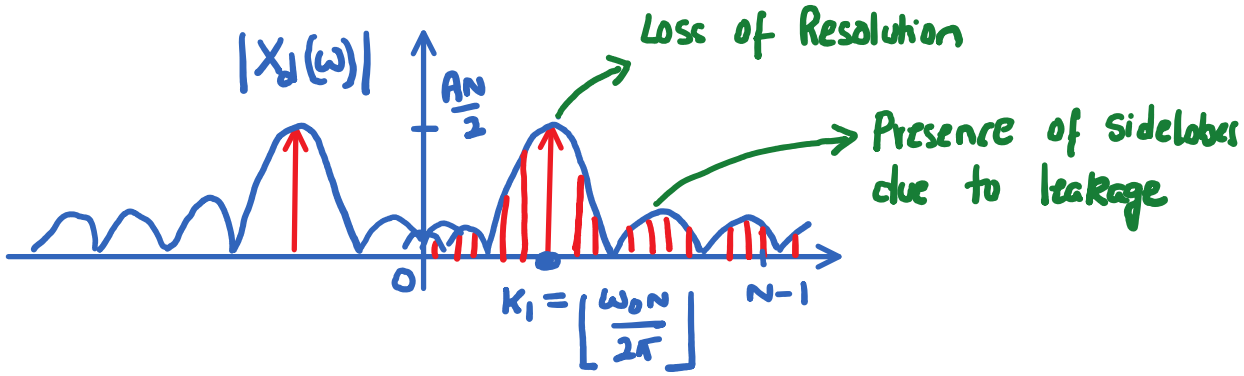


Figure 1.2. Implications of windowing

Recall that the DFT is a sampled version of the DTFT. Hence a windowed cosine function can have many non-zero spectral lines as shown below:



Note:

- DFT of windowed cosine can have several non-zero values.
- For a signal that is a sum of cosines one can have multiple peaks that may be indistinguishable.
- Recall the relationship between normalized discrete-time frequency variable ω and continous-time frequency variable .

$$\boldsymbol{\omega_0 = \Omega_0 T.}$$

Hence the **N** −length DFT of a sampled CT sinusoid, $x(t) = \cos(\Omega_0 t)$ will exhibit a peak at,

$$k = \left\lfloor \frac{\Omega_0 TN}{2\pi} \right\rceil$$

- **Can we do something to ensure that the DFT of a sinusoid has only two non-zero spectral lines in $(0, N-1)$? When will that be useful/not useful?**

5. Windowing Functions

The effect of windowing is that we lose resolution and there is spectral leakage caused by the strong discontinuity introduced by the rectangular window. Spectral leakage can be an issue while testing the performance of A/D converters since one may not be able to place the signal in a bin perfectly. IN that case the spectral leakage phenomenon can make the ADC appear much worse than it actually is. A solution to this problem is to have windows that decay to zero in a "smoother" fashion. While this causes the sidelobes to have smaller amplitude, the width of the main lobe increases. The figure below shows the spectral characteristics of commonly used windows in circuit design. You will be verifying these characteristics by performing Python based simulations.

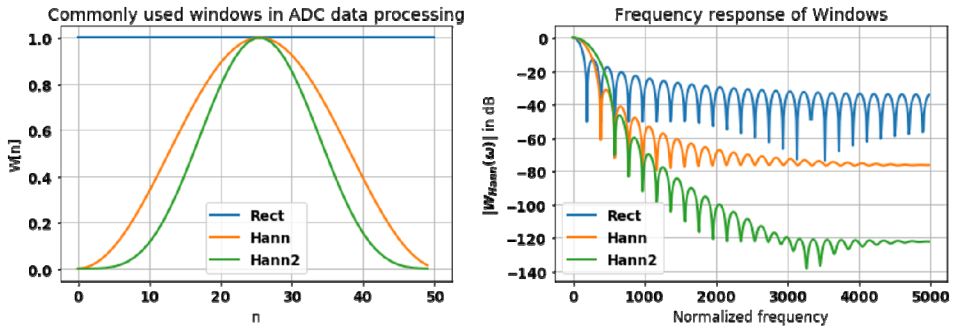


Fig. 1.3. Commonly used Windows.

6. SNR/SQNR Computation using DFT

Signal to Quantization noise ratio (SQNR) is a parameter based on which the performance of an ADC is evaluated. This quantity is evaluated using DFT by making use of the Parseval's relation,

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Where, $\mathbf{x[n]}$ denotes the DT N −length signal and $\mathbf{X[k], k = 0, 1, \dots, N-1}$ is the DFT of $\mathbf{x[n]}$.

Consider a sine wave input given by,

$$\mathbf{x_a(t) = Asin(\Omega_0 t)}.$$

Assume that the signal is sampled with sampling period $\mathbf{T}$ and then quantized as shown below (we will discuss this in detail in the following lecture).



The process of analog to digital conversion introduces quantization noise. Let the sampled signal $\mathbf{x[n]}$ be,

$$\mathbf{x[n] = Asin(\omega_o n) + q[n]},$$

where, $\boldsymbol{\omega_0 = \Omega_0 T}$ and the samples of quantization noise $\mathbf{q[n]}$ are uncorrelated.

Let $\mathbf{X[k]}$ and $\mathbf{Q[k], k = 0, 1, \dots, N-1}$ denote an $\mathbf{N}$ (even) length DFT of $\mathbf{x[n]}$. Let us assume that we have an integer $\mathbf{0 \le k_0 \le N-1}$ such that,

$$\boldsymbol{\omega_0 \approx \frac{2\pi k_0}{N}}.$$

An approximate measure of the SQNR is then given by,

$$SQNR = \frac{|X[k_0]|^2}{\sum_{\substack{p=0 \\ p \neq k_0}}^{N/2-1} |Q[k]|^2}$$