Ex

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta)$$

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Steps

1) State Space	$x_1 = \theta$ $x_2 = \dot{\theta}$ $\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{g}{l}\sin(x_1)$
2) Find Equilibrium points $\frac{d\underline{x_e}}{dt} = \underline{0}$	$\frac{dx_{1e}}{dt} = 0 \qquad \frac{dx_{2e}}{dt} = -\frac{g}{l}\sin(x_1) = 0$ $\underline{x_e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \underline{x_e} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

3) Examine small displacements from equilibrium

$$\underline{x} = \underline{x_e} + \underline{\Delta x}$$

small displacement

$$\frac{d\underline{x}}{dt} = \frac{d\underline{x_e}}{dt} + \frac{d\underline{\Delta x}}{dt}$$

$$\frac{d\underline{x}}{dt} = \frac{d\underline{\Delta x}}{dt}$$

4) Taylor series expand $\underline{f}$ around $\underline{x_e}$

$$f_1(\underline{x}) = f_1(\underline{x_e} + \underline{\Delta x}) \approx f_1(\underline{x_e}) + \Delta x_1 \frac{\partial f_1}{\partial x_1}\Big|_{x_e} + \Delta x_2 \frac{\partial f_1}{\partial x_2}\Big|_{x_e} + \dots + \Delta x_n \frac{\partial f_1}{\partial x_n}\Big|_{x_e}$$

* Only keep the Δx terms
neglect $O(\Delta x^2)$

$$\underline{f}(\underline{x_e} + \underline{\Delta x}) = \underbrace{\cancel{\underline{f}(\underline{x_e})}}_{0} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{\text{Jacobian}}\Bigg|_{\underline{x_e}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$\underline{f}(\underline{x_e} + \underline{\Delta x}) \approx A\big|_{x_e} \underline{\Delta x}$$

5) Linearized: $$\frac{d\underline{\Delta x}}{dt} = A\big|_{\underline{x_e}} \underline{\Delta x}$$

Ex) $$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{g}{l}\sin(x_1)$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & 0 \end{bmatrix}$$

$$\underline{x_e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\big|_{\underline{x_e}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \Delta x_2 \\ -\frac{g}{l}\Delta x_1 \end{bmatrix}$$

$$\underline{x_e} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$A\big|_{x_e} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \Delta x_2 \\ \frac{g}{l}\Delta x_1 \end{bmatrix}$$

$$\underline{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underline{x}_e = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \Delta x_2 \\ -\frac{g}{l}\Delta x_1 \end{bmatrix} \qquad \frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \Delta x_2 \\ \frac{g}{l}\Delta x_1 \end{bmatrix}$$

$$\underline{\Delta x}_0 = \begin{bmatrix} \Delta x_{10} \\ \Delta x_{20} \end{bmatrix} = \begin{bmatrix} \Delta x_{10} \\ 0 \end{bmatrix} \longleftrightarrow \text{Same IC's}$$

$$\boxed{\Delta x_1(t) = \Delta x_{10} \cos\left(t\sqrt{\frac{g}{l}}\right)} \qquad \boxed{\Delta x_1(t) = \Delta x_{10} \cosh\left(t\sqrt{\frac{g}{l}}\right)}$$

$$\frac{dx}{dt} = x(1-y) \qquad x_1 = x$$

$$x_2 = y$$

$$\frac{dy}{dt} = y(x-1)$$

$$\frac{dx_1}{dt} = x_1 - x_1 x_2$$

$$\frac{dx_2}{dt} = x_1 x_2 - x_2$$

Equilibrium:

$$0 = x_1 - x_1 x_2$$

$$0 = x_1 x_2 - x_2$$

$$\underline{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For this problem in particular this is trivial.

$$A = \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$$

$$A\big|_{x_e} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -\Delta x_2 \\ \Delta x_1 \end{bmatrix}$$

$$\Delta x_1 = \Delta x_{10} \cos(t) - \Delta x_{20} \sin(t)$$

$$\Delta x_2 = \Delta x_{10} \sin(t) + \Delta x_{20} \cos(t)$$