ECE 333

Lecture 10

Betz Limit, TSR, Turbine Power Curves

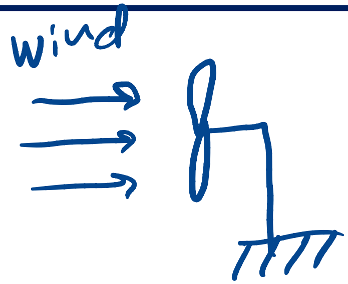
Jonathon Schuh

Slides Provided by Olga Mironenko

Department of Electrical and Computer Engineering

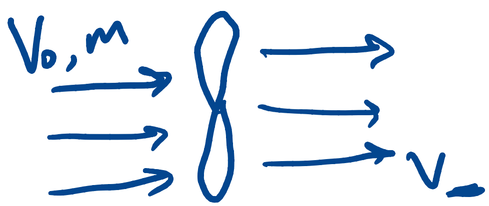
Fall 2024

Can we capture all kinetic energy in the wind?



* If we do it, $V_{exit} = 0$

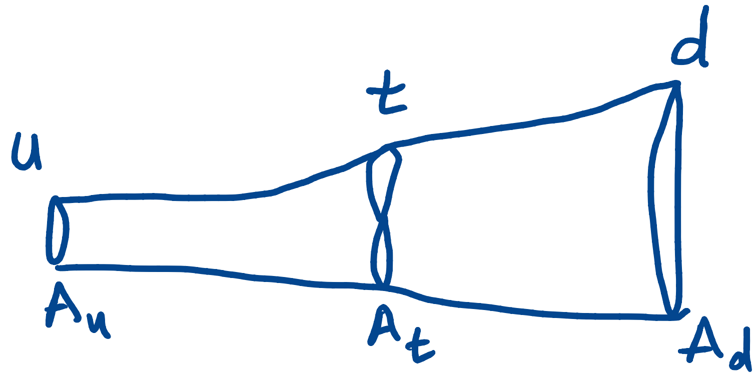
— Can't.



$V_- < V_0$, won't be 0.

Can we capture all kinetic energy in the wind?

upstream: u
downstream: d
turbine: t



if: $V_u = V_d$, no energy harvesting

$$0 < V_d < V_t < V_u$$

Conservation of mass

$$\frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

$$\dot{m}_{in} - \dot{m}_{out} = 0$$

$\frac{dm}{dt} = 0$ for no mass accumulation

$\rho A dx$

$\dot{m} = \rho A \frac{dx}{dt}$

$\dot{m}_{in} = \rho_{in} A_{in} V_{in}$

$\underbrace{\rho_{in} A_{in} V_{in}}_{\text{mass flow rate}} = \rho_{out} A_{out} V_{out}$

$\dot{m}_{out} = \rho_{out} A_{out} V_{out}$

if ρ is constant

$\underbrace{A_{in} V_{in}}_{\text{volumetric flow rate}} = A_{out} V_{out}$

$$\frac{dm}{dt} = \frac{d}{dt}\left(\int_{\Omega} \rho \, d\forall\right)$$

$$= \underbrace{\int_{\Omega} \frac{\partial \rho}{\partial t} \, d\forall}_{\text{Accumulation}} + \underbrace{\int_{\partial\Omega} \rho \underline{V} \cdot \hat{n} \, dA}_{\text{mass flow rate}}$$

$$\Rightarrow \quad \rho_{in} A_{in} V_{in} = \rho_{out} A_{out} V_{out}$$

Bernoulli's principle

– Restatement of conservation of Energy

$$W = \vec{F} \cdot \vec{x} = Fx$$

$$\Delta KE = W_{on}$$

$$= P_1 A_1 dx_1 - P_2 A_2 dx_2 + W_g$$

$$= P_1 A_1 (v_1 dt) - P_2 A_2 (v_2 dt) + (-\Delta PE)$$

$$= P_1 A_1 v_1 dt - P_2 A_2 v_2 dt + (gmz_1 - gmz_2)$$

$$= P_1 A_1 v_1 dt - P_2 A_2 V_2 dt + \rho_1 g A_1 v_1 dt - \rho_2 g A_2 V_2 dt$$

$$= P_1 A_1 V_1 dt - P_2 A_2 V_2 dt + \rho_1 g A_1 V_1 dt - \rho_2 A_2 g V_2 dt$$

$$\frac{1}{2}(\rho_2 A_2 V_2 dt) V_2^2 \ \frac{1}{2}(\rho_1 A_1 V_1 dt) V_1^2 = P_1 A_1 V_1 dt - P_2 A_2 V_2 dt + \rho_1 g A_1 V_1 dt - \rho_2 A_2 g V_2 dt$$

$$\frac{1}{2}\rho_1 A_1 V_1 d\cancel{t}\, V_1^2 + \rho_1 A_1 V_1 d\cancel{t}\, g z_1 + P_1 A_1 V_1 d\cancel{t} = \frac{1}{2}\rho_2 A_2 V_2 d\cancel{t}\, V_2^2 + \rho_2 A_2 V_2 d\cancel{t}\, g z_2 + P_2 A_2 V_2 d\cancel{t}$$

* No loss in energy

* Viscosity is negligible

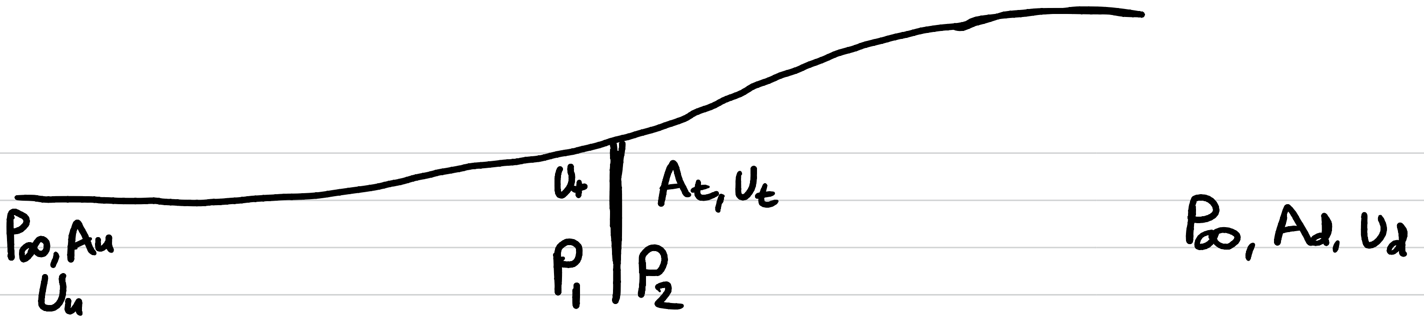
* Flow is incompressible ($Ma < 0.3$)

* Points 1 and 2 are on same streamline in same flow field

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \Rightarrow \frac{\rho_2 A_2 V_2}{\rho_1 A_1 V_1} = 1$$

$$\boxed{\frac{V_1^2}{2} + g z_1 + \frac{P_1}{\rho_1} = \frac{V_2^2}{2} + g z_2 + \frac{P_2}{\rho_2}}$$

Bernoulli's Equation



ρ is constant

$$A_u U_u = A_t U_t = A_d U_d$$

Change in Momentum at Turbine

$$F = \dot{m}\Delta V$$

$$(P_1 - P_2)A_t = (\rho A_t U_t)(U_u - U_d)$$

$$\boxed{P_1 - P_2 = \rho U_t (U_u - U_d)}$$

Bernoulli

$$P_\infty + \frac{1}{2}\rho U_u^2 + g\cancel{z_u} = P_1 + \frac{1}{2}\rho U_t^2 + g\cancel{z_t}$$

$$P_\infty + \frac{1}{2}\rho U_u^2 = P_1 + \frac{1}{2}\rho U_t^2$$

$$P_\infty + \frac{1}{2}\rho U_d^2 = P_2 + \frac{1}{2}\rho U_t^2$$

$$-\left(P_\infty + \frac{1}{2}\rho U_d^2 = P_2 + \frac{1}{2}\rho U_t^2\right)$$

$$\boxed{P_1 - P_2 = \frac{1}{2}\rho\left(U_u^2 - U_d^2\right)}$$

$$\frac{1}{2}\rho\left(U_u^2 - U_d^2\right) = \rho U_t (U_u - U_d)$$

$$\frac{1}{2}\rho(U_u + U_d)(U_u - U_d) = \rho U_t (U_u - U_d)$$

$$\boxed{U_t = \frac{U_u + U_d}{2}}$$

$$\eta = \frac{P_{out}}{P_{in}}$$

$$P_{in} = \tfrac{1}{2}\rho A U^3$$
$$= \tfrac{1}{2}\rho A_t U_t^3$$

<u>Assume:</u> $U_t = U_u$

$$\boxed{P_{in} = \tfrac{1}{2}\rho A_t U_u^3}$$

$$P_{out} = (P_1 - P_2) A_t U_t$$
$$= \rho A_t U_t (U_u - U_d) U_t$$
$$= \rho A_t U_t^2 (U_u - U_d)$$
$$= \rho A_t \left(\frac{U_u + U_d}{2}\right)^2 (U_u - U_d)$$
$$= \frac{1}{4}\rho A_t (U_u + U_d)(U_u + U_d)(U_u - U_d)$$
$$= \frac{1}{4}\rho A_t (U_u + U_d)(U_u^2 - U_d^2)$$
$$= \frac{1}{4}\rho A_t \left(U_u\left[1 + \frac{U_d}{U_u}\right]\right)\left(U_u^2\left[1 - \left[\frac{U_d}{U_u}\right]^2\right]\right)$$
$$= \frac{1}{4}\rho A_t U_u^3 \left(1 + \frac{U_d}{U_u}\right)\left(1 - \left(\frac{U_d}{U_u}\right)^2\right)$$

$$b = \frac{U_d}{U_u} \quad \Rightarrow \quad \boxed{P_{out} = \frac{1}{4}\rho A_t U_u^3 (1+b)(1-b^2)}$$

$$\eta = \frac{\frac{1}{4}\cancel{\rho A_t U_u^3}(1+b)(1-b^2)}{\frac{1}{2}\cancel{\rho A_t U_u^3}}$$

$$\boxed{\eta = \tfrac{1}{2}(1+b)(1-b^2)}, \qquad \eta = \tfrac{1}{2}(1 - b^2 + b - b^3)$$

<u>Max η:</u> $\frac{d\eta}{db} = 0 \Rightarrow 0 = \tfrac{1}{2}(-2b + 1 - 3b^2)$

$$3b^2 + 2b - 1 = 0$$

$$b = \frac{-2 \pm \sqrt{4 - 4(3)(-1)}}{2(3)} \Rightarrow b = \frac{-2 \pm 4}{6}$$

$$b = -1, \tfrac{1}{3}$$

if $b = \frac{1}{3}$, $\eta = \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^2 + \frac{1}{3} - \left(\frac{1}{3}\right)^3\right)$

$$\boxed{\eta = \frac{16}{27} \approx 59.26\%}$$

Conservation of mass

Conservation of energy

Betz Efficiency

Tip Speed Ratio (TSR)

- Actual rotor efficiency will be less than the Betz limit
- For a given wind speed, efficiency is a function of the rotor rotational speed
  - Spin too slowly, wind passes by without being "captured"
  - Spin too quickly, blades cause wind turbulence which reduces the efficiency of the blades.
- Rotor efficiency can be expressed as a function of its <u>Tip Speed Ratio</u>:

meters

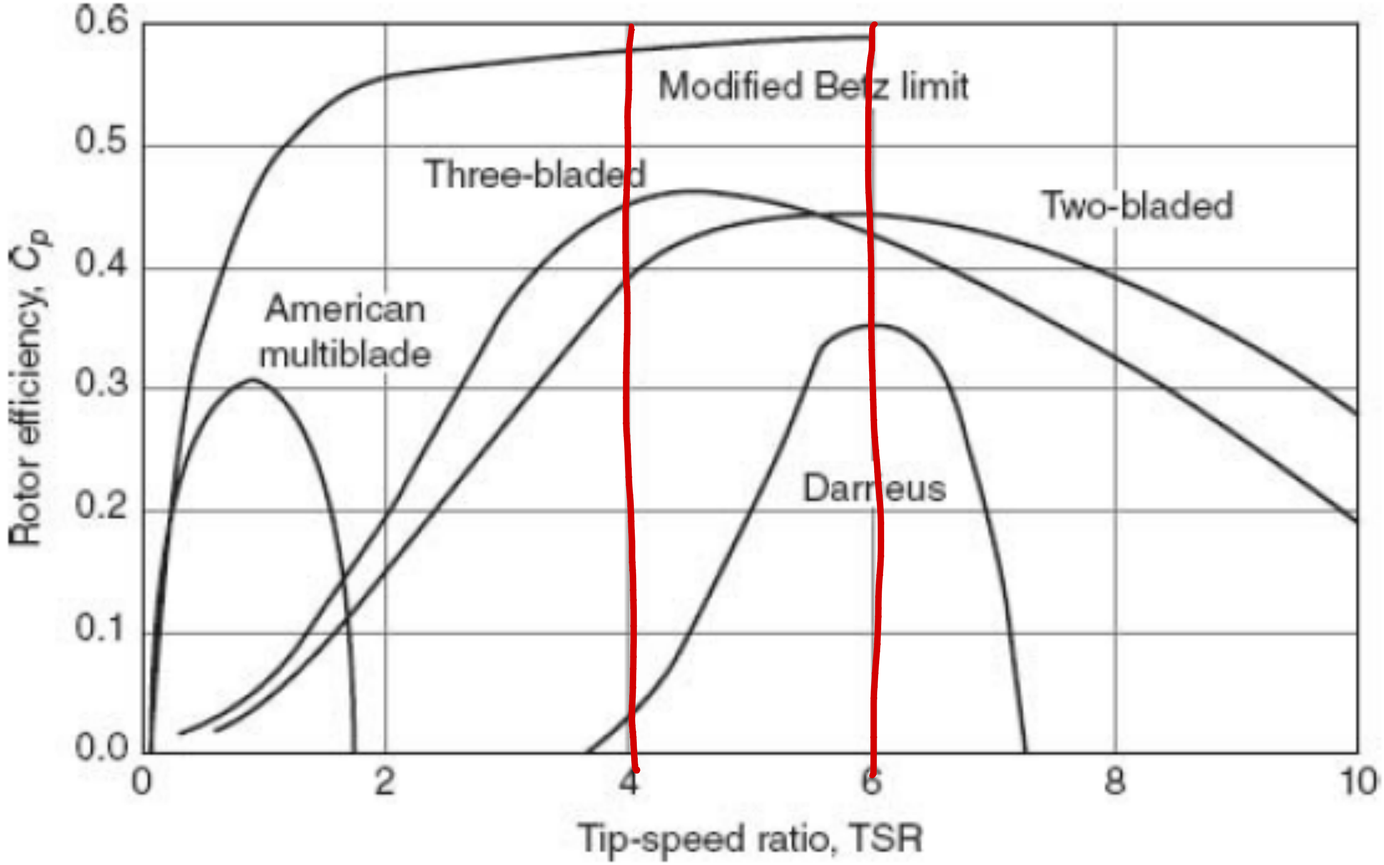
$$\text{Tip speed ratio (TSR)} = \frac{\text{Rotor tip speed}}{\text{Upwind wind speed}} = \frac{\text{rpm} \times \pi D}{60v} \qquad (7.30)$$

wind speed

Tip Speed Ratio

- Optimal TSR for 2- and 3-blade modern turbines is in the 4-6 range.

Fig. 7.18



Idealized Power Curve

Electric machines are limited by their power rating.

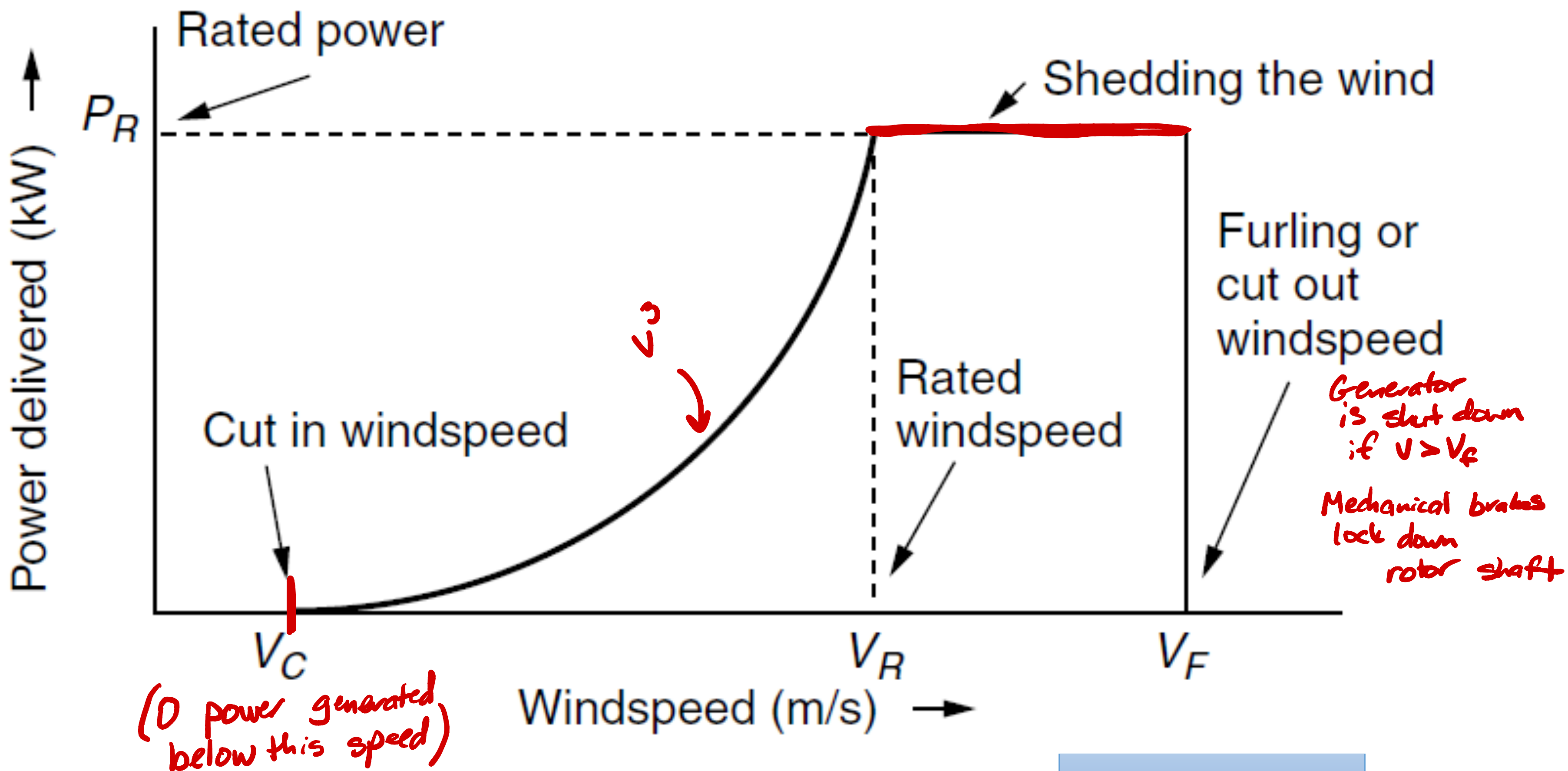


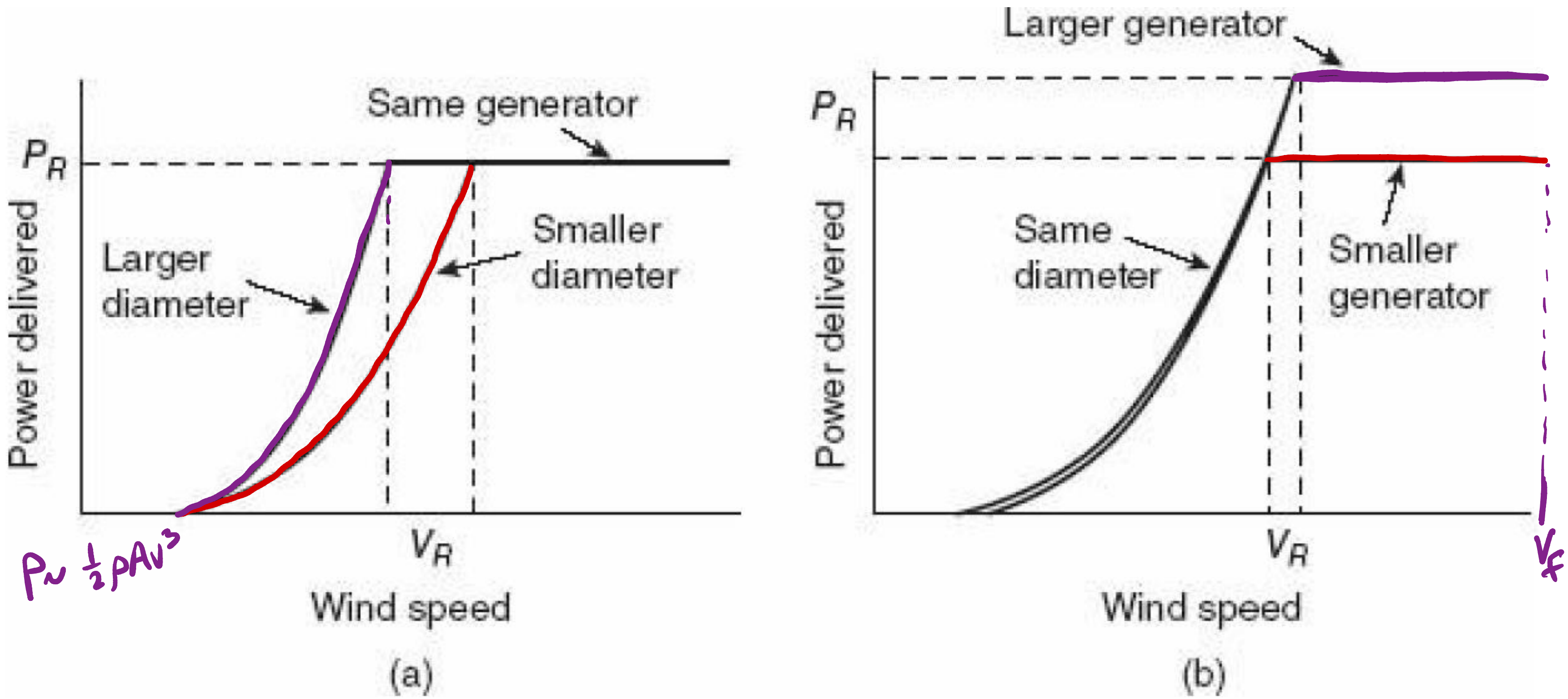
Figure 7.19

Impact of design parameters

- Tradeoffs between rotor diameter and generator size (power rating)

Figure 7.20

Allows Turbine to operate at higher speeds



Real Power Curves

- First number: Power rating
- Second number: Rotor diameter

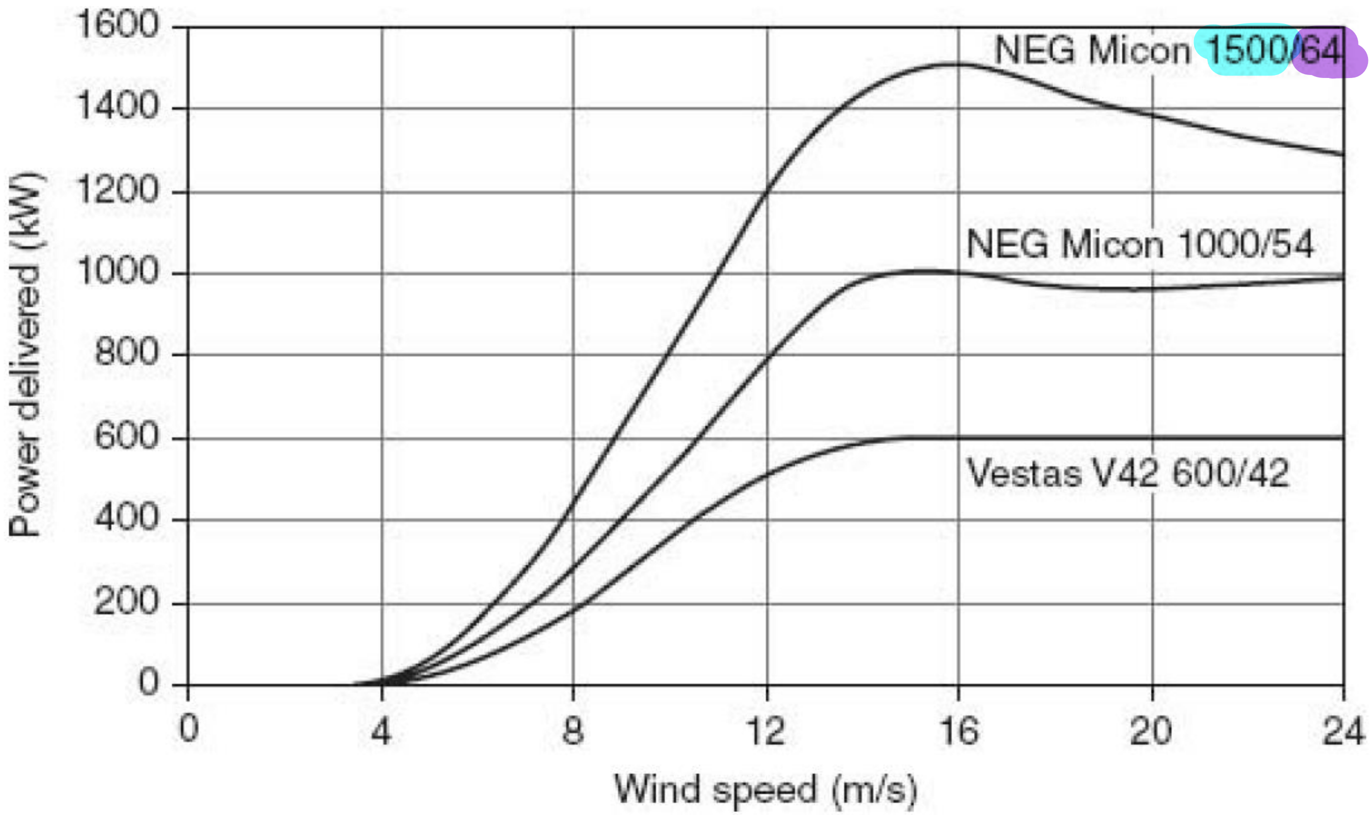


Figure 7.21

ECE 333

Lecture 11

Average Power in the Wind
Estimating Wind Turbine Energy Production

Jonathon Schuh

Slides Provided by Olga Mironenko

Department of Electrical and
Computer Engineering

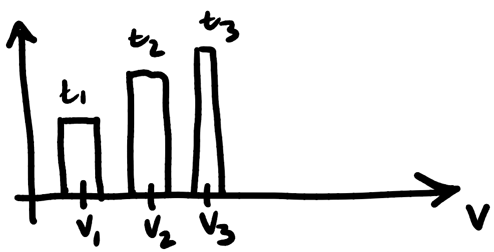
Discrete Wind Histogram

On average, how much energy could we expect to get out of the turbine?

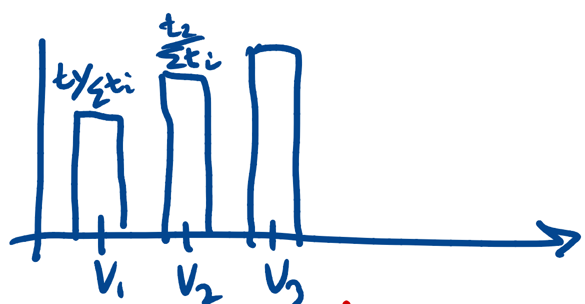
$$\langle P_{wind} \rangle = \langle \tfrac{1}{2} \rho A v^3 \rangle = \tfrac{1}{2} \rho A \langle v^3 \rangle$$

time average

$$\langle v^3 \rangle \neq \langle v \rangle^3$$



t_i: # of Hours at windspeed v_i



$$\frac{t_i}{\sum t_i} < 1$$

$$\sum \frac{t_i}{\sum t_i} = 1$$

Normalized Histogram

* Probability Mass Function (PMF)

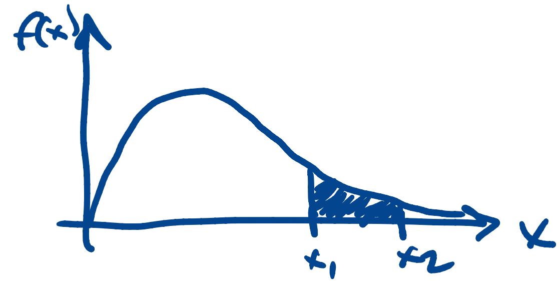
Wind Speed Probability Density Function

* Wind speed is continuous

Continuous Random Variable: probability density function (PDF)

let x continuous random variable

It's PDF $f(x)$



$f(x) \geq 0 \quad \forall x$

$\int_{-\infty}^{\infty} f(x)dx = 1$ (area under curve = 1)

Probability x is between x_1 and x_2

$$P = \int_{x_1}^{x_2} f(x)dx$$

If $x_1 \leq x \leq x_1 + \delta \quad \delta \ll 1$

$$p = f(x_1) \cdot \delta$$

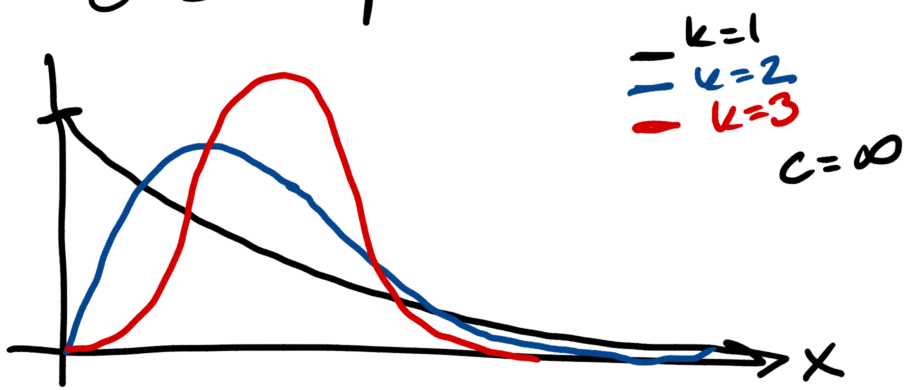
$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx$$

Weibull and Rayleigh Statistics

Weibull

$$f(x) = \frac{k}{c}\left(\frac{x}{c}\right)^{k-1} e^{-\left(\frac{x}{c}\right)^k} \qquad x > 0$$

k: shape parameter
c: scale parameter



For k=2: Rayleigh PDF

$$f(x) = \frac{2x}{c^2} e^{-\left(\frac{x}{c}\right)^2}$$

$$\langle x \rangle = \int_0^\infty x f(x)\,dx = \int_0^\infty 2\left(\frac{x}{c}\right)^2 e^{-\left(\frac{x}{c}\right)^2} dx$$

$$\langle x \rangle = \frac{\sqrt{\pi}}{2} c \quad \Rightarrow \quad c = \frac{2}{\sqrt{\pi}} \langle x \rangle$$

Average Power in the Wind with Rayleigh Statistics

$$f(x) = \frac{2x}{\left(\frac{2}{\sqrt{\pi}}\langle x\rangle\right)^2} e^{-\left(\frac{x}{\frac{2}{\sqrt{\pi}}\langle x\rangle}\right)^2}$$

$$= \frac{\pi}{2}\left(\frac{x}{\langle x\rangle^2}\right) e^{-\frac{\pi}{4}\left(\frac{x}{\langle x\rangle}\right)^2}$$

$$P_{wind} = \frac{1}{2}\rho A v^3$$

$$\langle P_{wind}\rangle = \int_0^\infty \frac{1}{2}\rho A v^3 \cdot f(v)\, dv$$

$$= \int_0^\infty \frac{1}{2}\rho A v^3 \left(\frac{\pi}{2}\frac{v}{\langle v\rangle^2} e^{-\frac{\pi}{4}\left(\frac{v}{\langle v\rangle}\right)^2}\right) dv$$

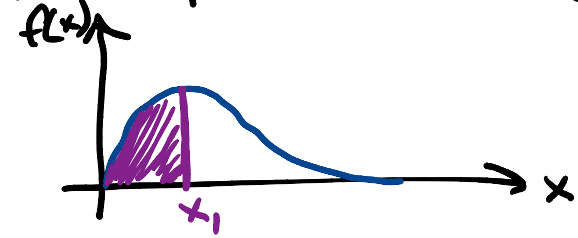
$$\boxed{\langle P_{wind}\rangle = \frac{6}{\pi}\left(\frac{1}{2}\rho A \langle v\rangle^3\right)}$$

Rayleigh Assumption

Estimating wind turbine energy production

Energy: power curve of the machine and a statistical model of the wind

Wind speed Cumulative Distribution Function (CDF)

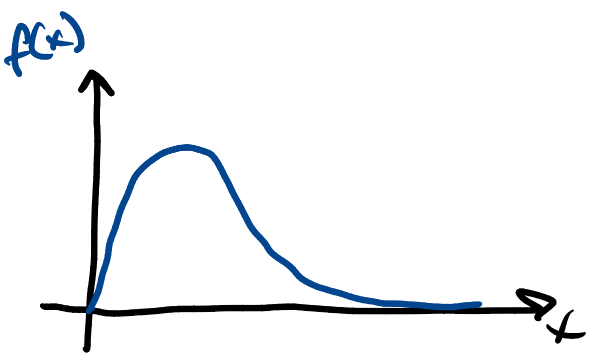


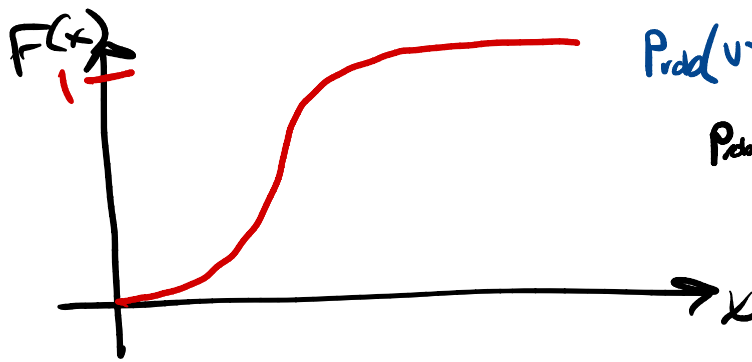
probability that $x \le x_1 = F(x) = \int_0^{x_1} f(x)\,dx$

Assume Rayleigh PDF

$$F(v) = \int_0^{v} \frac{\pi v'}{2\langle v\rangle^2} e^{-\frac{\pi}{4}\left(\frac{v'}{\langle v\rangle}\right)^2} dv'$$

$$F(v) = 1 - e^{-\frac{\pi}{4}\left(\frac{v}{\langle v\rangle}\right)^2}$$





$Prob(v>v_1) = 1 - Prob(v \le v_1)$

$Prob(v>v_1) = e^{-\frac{\pi}{4}\left(\frac{v_1}{\langle v\rangle}\right)^2}$

Example (7.7 from Masters)

2.1 MW $\quad V_c = 3.7\ m/s \quad V_R = 11\ m/s \quad V_F = 20\ m/s$

$\langle v \rangle = 7\ m/s$

a) $V < V_c$

$V < 3.7\ m/s$

$$\text{Prob}(V < 3.7) = 1 - e^{-\frac{\pi}{4}\left(\frac{3.7}{7}\right)^2}$$

$$= 0.197$$

$0.197(8700\ hr/year)$

$1714.1\ hrs/year$

~ 2.4 months

b) $V > V_F$

$$\text{Prob}(V > 20) = 1 - \text{Prob}(V < 20)$$

$$= 1 - F(20)$$

$$= e^{-\frac{\pi}{4}\left(\frac{20}{7}\right)^2} = 0.00164$$

$\#\ hrs = 0.00164(8700)$

$= \boxed{14.4\ hrs/year}$

Example (7.7 from Masters)