Synchronous machines.

Consists of a stator (a static part) and a rotor (a rotating part).

Both stator and rotor has windings or coils carrying currents.

Thumb rule } # coils/windings on Stator = # phases of the machine.

Agenda:

① Derive $\underline{\lambda} = L(\theta)\,\underline{i}$ Inductance of the coils in a matrix

Vectors with as many elements as the # coils.

⚠ $\underline{\lambda} = L(\theta)\,\underline{i} \Rightarrow$ Circuit is electrically linear.

② Derive co-energy $W_m'(\underline{i}, \theta) = \frac{1}{2}\,\underline{i}^T L(\theta)\,\underline{i}$

③ Derive torque of electrical origin

$$T^e(\underline{i}, \theta) = \frac{\partial W_m}{\partial \theta}.$$

④ Derive mechanical equation

$$J\ddot{\theta} = \underbrace{T^e}_{} + \underbrace{T^m}_{} - \underbrace{T^f}_{}$$

torque of electrical origin

torque from a governor

some mechanical system that drives or rotates the rotor.

Think of a turbine in a hydroelectric generator.

torque due to friction

We shall mostly concentrate on frictionless systems, where $T^f = 0$.

⑤ Derive electrical equation

If the currents $\underline{i}$ are into the system,

$$\underbrace{v_{coil}}_{} = \underbrace{i_{coil}}_{}\ \underbrace{R_{coil}}_{} + \frac{d\underbrace{\lambda_{coil}}_{}}{dt}$$

voltage across any of the coils

current in the corresponding coil

resistance in that coil

flux linkage in that coil.

(6) Understand steady-state operation with sinusoidal currents

$\ddot{\theta} = 0$. No angular accelaration

$\Rightarrow \dot{\theta} = \omega_s$ (A synchronous frequency is maintained throughout.)

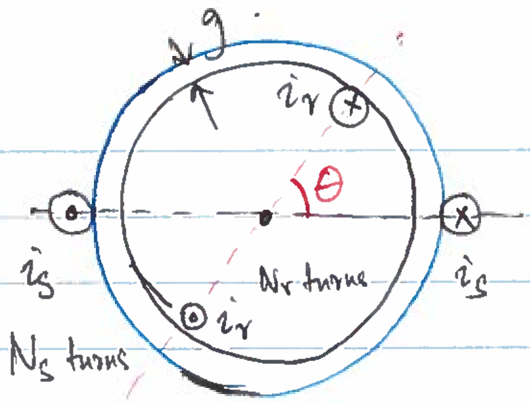
What do we want to compute in steady-state?

(1) Compute the mechanical power input required to maintain steady-state.

(2) Compute an equivalent circuit for the synchronous machine.

Today's class
- Assume we know $\lambda = L(\theta) i$ for 1ϕ, and 2ϕ machines.
- Compute mechanical power input at steady-state

Next class
1. How to derive $\lambda = L(\theta) i$ for 1ϕ, 2ϕ machines, and even 3ϕ round-rotor machines.
2. Find equivalent circuits for 2ϕ machines.



Single-phase machines.

In this class, we shall not derive the $\lambda = L(\theta)\, i$ description

But, we did derive it in an earlier lecture.

$$\begin{cases} \lambda_s = N_s^2 L_0 i_s + N_s N_r L_0 \left(1 - \dfrac{|2\theta|}{\pi}\right) i_r \\ \lambda_r = N_s N_r L_0 \left(1 - \dfrac{|2\theta|}{\pi}\right) i_s + N_r^2 L_0 i_r . \end{cases}$$

The $|\cdot|$ sign can be shown by considering $-$ve values of θ.

"approximately equal to"

$$\lambda_s = L_s i_s + M\cos\theta \; i_r$$

$$\lambda_r = M\cos\theta \; i_s + L_r i_r ,$$

where we have folded L_0, L_r, N_s, N_r into our new constants L_s, L_r, M.

Remark: The coils can be distributed over the stator/rotor to make this approximation exact.

- Computing $W_m'(i_s, i_r, \theta)$.

$$W_m'(i_s, i_r, \theta) = \frac{1}{2}\begin{pmatrix} i_s \\ i_r \end{pmatrix}^T \begin{pmatrix} L_s & M\cos\theta \\ M\cos\theta & L_r \end{pmatrix}\begin{pmatrix} i_s \\ i_r \end{pmatrix}$$

$$= \frac{1}{2} L_s i_s^2 + \frac{1}{2} L_r i_r^2 + M\cos\theta \; i_s i_r$$

- $T^e(i_s, i_r, \theta) = \frac{\partial W_m'}{\partial \theta} = -M \sin\theta \cdot i_s\, i_r$.

- Mechanical equation

$$J\ddot{\theta} = T^e + T^m - \underbrace{T^f}$$

assume frictionless
$\Rightarrow T^f = 0$.

$$= -M\sin\theta\, i_s\, i_r + \underbrace{T^m}$$

torque from mechanical governor

- Steady-state w/ sinusoidal currents.

$$\begin{cases} i_s = I_s \cos(\omega_s t), \quad i_r = I_r \cos(\omega_r t) \\ \ddot{\theta} = 0, \text{ and } \quad \theta = \omega_m t + \gamma. \end{cases}$$

$$\Rightarrow \quad T^m = M \sin(\omega_m t + \gamma)\, I_s \cos(\omega_s t)\, I_r \cos(\omega_r t).$$

$$= \frac{1}{4} M I_s I_r \left[\begin{array}{l} \sin(\omega_1 t + \gamma) + \sin(\omega_2 t + \gamma) \\ + \sin(\omega_3 t + \gamma) + \sin(\omega_4 t + \gamma) \end{array} \right].$$

where
$$\begin{aligned} \omega_1 &= \omega_m + \omega_s - \omega_r, \\ \omega_2 &= \omega_m - \omega_s + \omega_r, \\ \omega_3 &= \omega_m + \omega_s + \omega_r, \\ \omega_4 &= \omega_m - \omega_s - \omega_r. \end{aligned}$$

Derivation on next page

Skip it on first reading of the notes

Messy derivation

$$\sin(\omega_m t+\gamma)\cos(\omega_s t)\cos(\omega_r t)$$

$$= \frac{1}{2}\sin(\omega_m t+\gamma)\Big[\cos(\omega_s+\omega_r)t + \cos(\omega_s-\omega_r)t\Big]$$

$$= \frac{1}{2}\Big[\sin(\omega_m t+\gamma)\cos(\omega_s+\omega_r)t\Big]$$

$$+ \frac{1}{2}\Big[\sin(\omega_m t+\gamma)\cos(\omega_s-\omega_r)t\Big].$$

$$= \frac{1}{4}\Big[\sin(\omega_m t+\gamma+\omega_s t+\omega_r t) + \sin(\omega_m t+\gamma-\omega_s t-\omega_r t)\Big]$$

$$+ \frac{1}{4}\Big[\sin(\omega_m t+\gamma+\omega_s t-\omega_r t) + \sin(\omega_m t+\gamma-\omega_s t+\omega_r t)\Big].$$

$$= \frac{1}{4}\Big[\sin\big(\underbrace{(\omega_m+\omega_s+\omega_r)}_{=\omega_3}t+\gamma\big)\Big]$$

$$+ \frac{1}{4}\Big[\sin\big(\underbrace{(\omega_m-\omega_s-\omega_r)}_{=\omega_4}t+\gamma\big)\Big]$$

$$+ \frac{1}{4}\Big[\sin\big(\underbrace{(\omega_m+\omega_s-\omega_r)}_{=\omega_1}t+\gamma\big)\Big]$$

$$+ \frac{1}{4}\Big[\sin\big(\underbrace{(\omega_m-\omega_s+\omega_r)}_{=\omega_2}t+\gamma\big)\Big].$$

• If $\omega_1 \neq 0$, $\omega_2 \neq 0$, $\omega_3 \neq 0$, $\omega_4 \neq 0$,

then $\underbrace{\langle T^m \rangle = 0}$.

If average torque $= 0$ in steady-state, then, so is average mechanical power input.

Can you make a machine act as a generator or motor if average mech. power $= 0$?

— NO!

$\Rightarrow$ $\omega_i \neq 0$ for $i = 1, 2, 3, 4$ is an uninteresting case.

• If one of the ω_i's are zero, then other ω_j's are not zero for $j \neq i$

For example,

$$\omega_1 = 0 \Rightarrow \omega_2 \neq 0, \ \omega_3 \neq 0, \ \omega_4 \neq 0.$$

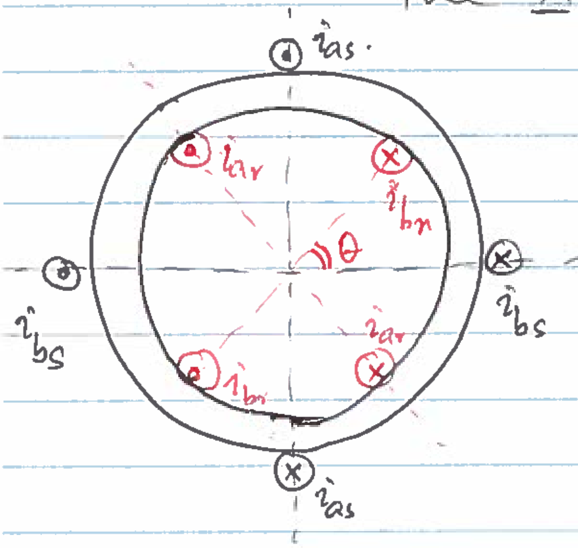
$\Rightarrow$ T^m has to be time-varying to operate machine in steady state!

— Difficult to produce time-varying torque

Some Additions

2-ϕ machines.

In the next class, we shall learn how the $\underline{\lambda} = L(\theta)\,\underline{i}$ relations are obtained!



In this class, we shall take them as given, and proceed to understanding steady-state operation with sinusoidal currents.

$$\underbrace{\begin{pmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{pmatrix}}_{=\underline{\lambda}} = \underbrace{\begin{pmatrix} L_s & 0 & M\cos\theta & -M\sin\theta \\ 0 & L_s & M\sin\theta & M\cos\theta \\ M\cos\theta & M\sin\theta & L_r & 0 \\ -M\sin\theta & M\cos\theta & 0 & L_r \end{pmatrix}}_{=L(\theta)} \underbrace{\begin{pmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{pmatrix}}_{=\underline{i}}$$

$$W_m'(i_{as}, i_{bs}, i_{ar}, i_{br}, \theta)$$

$$= \frac{1}{2}\underline{i}^T L(\theta)\,\underline{i}$$

$$= \frac{1}{2}L_s i_{as}^2 + \frac{1}{2}L_s i_{bs}^2 + \frac{1}{2}L_r i_{ar}^2 + \frac{1}{2}L_r i_{br}^2$$

$$+ M\cos\theta\,(i_{as} i_{ar} + i_{bs} i_{br}) + M\sin\theta\,(i_{bs} i_{ar} - i_{as} i_{br}).$$

- $T^e = \dfrac{\partial W_m'}{\partial \theta}$

$$= -M\sin\theta\left(i_{as} i_{ar} + i_{bs} i_{br}\right) + M\cos\theta\left(i_{bs} i_{ar} - i_{as} i_{br}\right).$$

- Mechanical equation:

$$J\ddot{\theta} = \underbrace{T^e}_{\text{we just calculated this.}} + \underbrace{T^m}_{\text{torque from mechanical governor}} - \underbrace{T^f}_{\text{assume frictionless} \Rightarrow T^f = 0.}$$

- Steady-state under sinusoidal currents

$$\ddot{\theta} = 0 \Rightarrow \dot{\theta} = \omega_m \Rightarrow \theta = \omega_m t + \gamma$$

$$i_{as} = I_s \cos(\omega_s t), \quad i_{bs} = I_s \sin(\omega_s t)$$

$$i_{ar} = I_r \cos(\omega_r t), \quad i_{br} = I_r \sin(\omega_r t)$$

- Also $T^m = -T^e$.

~~Average mechanical power input (P^m)~~

It turns out that $T^m = M I_s I_r \sin\left[(\omega_m - \omega_s + \omega_r)t + \gamma\right]$

Derivation is on the next page. Skip it on first reading!

$$T^m = -T^e$$

$$= M I_s I_r \left[\overbrace{+\sin(\omega_m t + \gamma)}^{=0} \cdot \left\{ \underbrace{\cos \omega_s t}_{\text{from } i_{as}} \underbrace{\cos \omega_r t}_{\text{from } i_{ar}} + \underbrace{\sin \omega_s t}_{\text{from } i_{bs}} \underbrace{\sin \omega_r t}_{\text{from } i_{br}} \right\} \right.$$

$$\left. - \cos(\omega_m t + \gamma) \left\{ \underbrace{\sin \omega_s t}_{\text{from } i_{bs}} \underbrace{\cos \omega_r t}_{\text{from } i_{ar}} - \underbrace{\cos \omega_s t}_{\text{from } i_{as}} \underbrace{\sin \omega_r t}_{\text{from } i_{br}} \right\} \right]$$

Use the formulae:

$$\cos\alpha \cos\beta + \sin\alpha \sin\beta = \cos(\alpha - \beta)$$

$$\sin\alpha \cos\beta - \cos\alpha \sin\beta = \sin(\alpha - \beta).$$

$$\therefore T^m = M I_s I_r \left[\sin(\omega_m t + \gamma) \cos(\omega_s t - \omega_r t) - \cos(\omega_m t + \gamma) \sin(\omega_s t - \omega_r t) \right].$$

$$= M I_s I_r \sin(\omega_m t + \gamma - \omega_s t + \omega_r t)$$

$$= M I_s I_r \sin\left[(\omega_m - \omega_s + \omega_r) t + \gamma \right]$$

What is the qualitative difference between T^e (or T^m) in 1-ϕ and 2-ϕ?

Think!

$$\langle T^m \rangle = \begin{cases} M I_s I_r \sin\gamma, & \text{if } \omega_m - \omega_s + \omega_r = 0, \\ 0, & \text{otherwise.} \end{cases}$$

What about instantaneous torque.

$$T^m = \begin{cases} M I_s I_r \sin\gamma, & \text{if } \omega_m - \omega_s + \omega_r = 0, \\ M I_s I_r \sin\left((\omega_m - \omega_s + \omega_r)t + \gamma\right), & \text{otherwise.} \end{cases}$$

Frequency condition: $\omega_m - \omega_s + \omega_r = 0$

$\Longrightarrow$ T^m (instantaneous torque required to maintain steady-state operation)

$= M I_s I_r \sin\gamma$ (a constant!).

Also, $\langle T^m \rangle = M I_s I_r \sin\gamma$, which is a non-zero torque.

$\Rightarrow$ Average mech. power input from the governor $\Big\} = T^m \omega_m$

$= M I_s I_r \sin\gamma \, \omega_m$

Last class, we studied the expressions for T^e in 1ϕ, 2ϕ machines.

Summary of calculations for 1ϕ machines for steady-state operations $\Rightarrow$ $\begin{cases} \text{when } \langle T^e \rangle \neq 0, \ T^e \text{ is time-varying} \\ \text{when } \langle T^e \rangle = 0, \text{ the machine is not of much use to us!} \end{cases}$

Summary of calculations for 2ϕ machines $\Rightarrow$ $T^e = \langle T^e \rangle = -M I_s I_r \sin\gamma$, when $i_{as} = I_s \cos\omega_s t$, $i_{bs} = I_s \sin\omega_s t$,

$i_{ar} = I_r \cos\omega_r t$, $i_{br} = I_r \sin\omega_r t$,

$\theta = \omega_m t + \gamma$,

$\omega_m = \omega_s - \omega_r$.

This class:- ① Derive equivalent circuit for 2ϕ-machines.

② Get a feel for how the $\lambda - i$ relationship are derived for 1ϕ, 2ϕ, 3ϕ machines.

2ϕ machines:-

tasks

① We begin with $\underline{\lambda} = L(\theta)\, \underline{i}$

② Write electrical equation

③ Manipulate it to get a circuit out of it.

$\lambda = L(\theta)\, i$ for 2ϕ machines look like :

Recall :

$$\begin{pmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{pmatrix} = \begin{pmatrix} L_s & 0 & M\cos\theta & -M\sin\theta \\ 0 & L_s & M\sin\theta & M\cos\theta \\ M\cos\theta & M\sin\theta & L_r & 0 \\ -M\sin\theta & M\cos\theta & 0 & L_r \end{pmatrix} \begin{pmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{pmatrix}.$$

$$\lambda_{as} = L_s\, i_{as} + M\cos\theta\, i_{ar} - M\sin\theta\, i_{br}.$$

Electrical equation :

$$\underbrace{v_{as}}_{\text{voltage across as coil}} = \underbrace{R_s}_{\text{resistance of the stator windings.}} i_{as} + \frac{d}{dt}\underbrace{\lambda_{as}}_{\text{flux linkage of as coil.}}$$

Assumptions: ① Steady-state operation satisfying frequency condition.

- $\theta = \omega_m t + \gamma$
- $\omega_m = \omega_s - \omega_r^{\nearrow =0}$
  $\Rightarrow \omega_m = \omega_s$
- $i_{as} = I_s \cos\omega_s t$
  $i_{bs} = I_s \sin\omega_s t$.

② $\omega_r = 0 \Rightarrow \begin{cases} i_{ar} = I_r \text{ (direct current)} \\ i_{br} = 0 \end{cases}$

Let's derive this !.

$$\cancel{v_{as} = R_s I_s \cos\omega_s t + \frac{d}{dt}\Big[L_s I_s \cos\omega_s t + M\cos(\omega_m t + \gamma) I_s \sin\omega_s t} - \cancel{M\cos\theta \cdot I_r}^{(\omega_m t+\gamma)}\Big].$$

$$v_{as} = R_s \underbrace{i_{as}}_{I_s\cos(\omega_s t)} + \frac{d}{dt}\left[L_s \underbrace{i_{as}}_{I_s\cos(\omega_s t)} + M\cos\underbrace{\theta}_{\omega_m t+\gamma,\ =\omega_s} \underbrace{i_{ar}}_{I_r} - M\sin\theta \underbrace{i_{br}}_{0}\right]$$

$$= R_s I_s \cos(\omega_s t) + \frac{d}{dt}\left[L_s I_s \cos\omega_s t\right]$$

$$+ \frac{d}{dt}\left[M I_r \cos(\omega_s t + \gamma)\right].$$

$$= R_s I_s \cos\omega_s t - \omega_s L_s I_s \sin\omega_s t - \omega_s M I_r \sin(\omega_s t + \gamma)$$

$$= \mathrm{Re}\left\{\sqrt{2}\cdot R_s \cdot \underbrace{\frac{I_s \angle 0}{\sqrt{2}}}_{=\bar{I}_{as}} \cdot e^{j\omega_s t}\right\}$$

$$+ \mathrm{Re}\left\{\sqrt{2}\cdot \underbrace{(j\omega_s L_s)}_{=jX_s} \cdot \underbrace{\frac{I_s}{\sqrt{2}}\angle 0}_{=\bar{I}_{as}} \cdot e^{j\omega_s t}\right\}.$$

$$+ \mathrm{Re}\left\{\sqrt{2}\cdot \underbrace{\boxed{\frac{\omega_s M I_r}{\sqrt{2}} \angle\left(\frac{\pi}{2}+\gamma\right)}}_{\bar{E}_{ar}} \cdot e^{j\omega_s t}\right\}.$$

Verify these 3 terms!

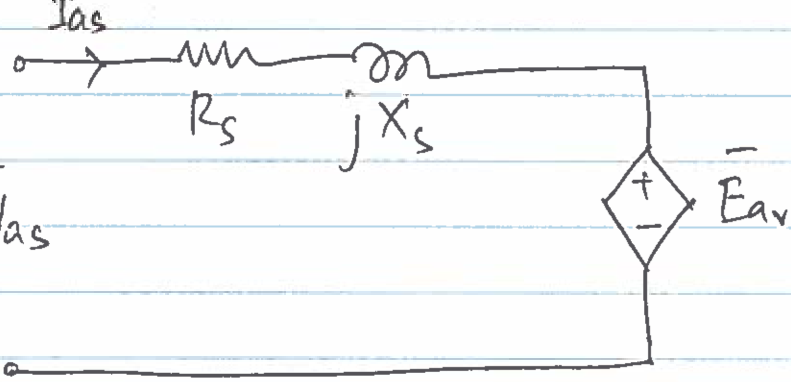
$$= \mathrm{Re}\left\{\sqrt{2}\cdot\left(R_s \bar{I}_{as} + jX_s \bar{I}_{as} + \bar{E}_{ar}\right)\cdot e^{j\omega_s t}\right\}.$$

Wait! The above looks like the time-domain signal for the phasor $R_s \bar{I}_{as} + jX_s \bar{I}_{as} + \bar{E}_{ar}$.

- Associate phasor to v_{as}: Call it $\bar{V}_{as}$

$$\therefore \bar{V}_{as} = R_s \bar{I}_{as} + jX_s \bar{I}_{as} + \bar{E}_{ar}$$

- Notice: this equation is the same as the relation between these quantities in the following circuit



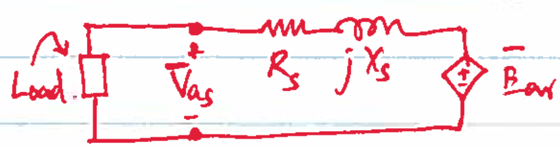
Equivalent circuit for a 2-ϕ synchronous machine.

Salient points:

-  = symbol for a <u>variable</u> voltage source

why variable? $|\bar{E}_{ar}|$ varies with I_r. the direct current into rotor windings

- If we connect stator phases to external circuit, say to a load, now we can analyze it by drawing the equiv. ckt.



Lot easier to analyze than the magnetic couplings.

Next topic:-

How are $\lambda = L(\theta)\, i$ relations derived in rotating machines?

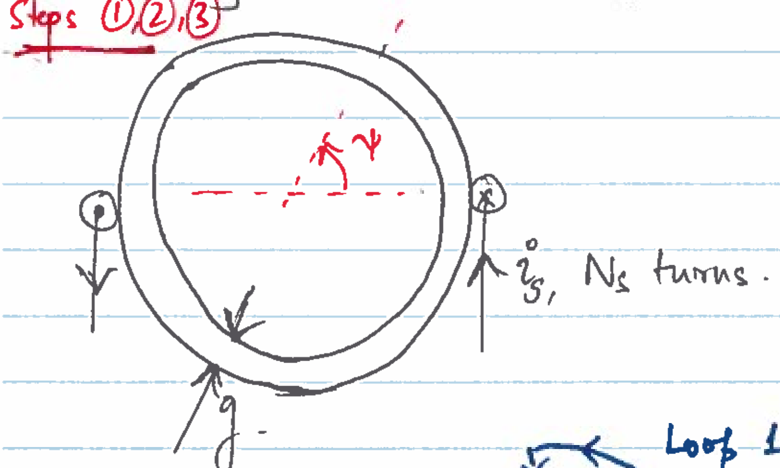
① Ampere's law
② Superposition
③ Sinusoidal approximation

$\Rightarrow$ yields $\vec{H}$, the magnetic field intensity in the air-gap.

④ $\vec{B} = \mu_0 \vec{H}$ in air-gap
⑤ Compute flux linkage using $\oint \vec{B} \cdot d\vec{a}$.

~~Achieving ①, ②, ③ for one set of~~

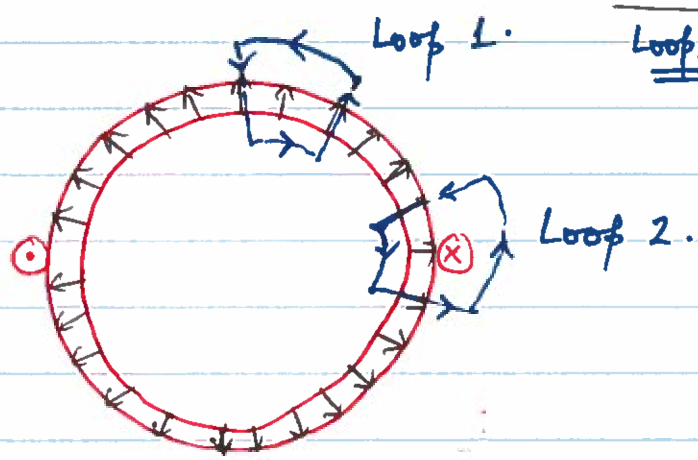
Steps ①, ②, ③



Standing assumptions:-

① $g \ll$ radius of rotor.

② $\vec{H}$ is radial in the air-gap.



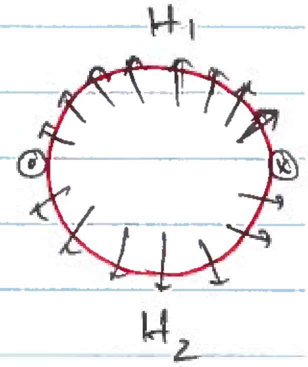
Loop1: Ampere's law around loop 1 yields $H(\psi) = \text{const.}$ for $\psi \in [0, \pi)$.

and $H(\psi) = \text{const.}$ for $\psi \in (\pi, 2\pi)$.

Do the details yourself!

Loop 2:- $\oint_{loop\,2} \vec{H}\cdot d\vec{l} =$ current coming "out" of loop 2.

$$\left\{ \begin{array}{llll} \text{Let} & H(\psi) = H_1 & \text{for} & 0 \leq \psi < \pi \\ \text{and} & H(\psi) = H_2 & \text{for} & \pi < \psi < 2\pi \end{array} \right\}$$



$$\therefore \oint_{Loop\,2} \bar{H}\cdot d\bar{l} = (H_2 - H_1)\cdot g$$

Current coming "out" of loop 2 $= -N_s i_s$.

$$\Rightarrow \quad H_2 - H_1 = \frac{-N_s i_s}{g}.$$

- **Claim:** $H_1 + H_2 = 0$ ← Can be argued from ① Symmetry ② Gauss' law ③ Right-hand rule

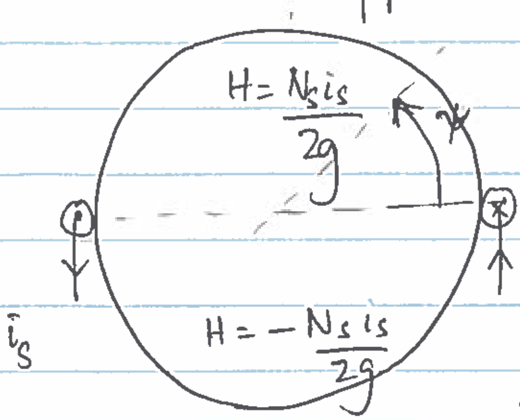
- Solve $H_2 - H_1 = -N_s i_s / g$ and $H_1 + H_2 = 0$

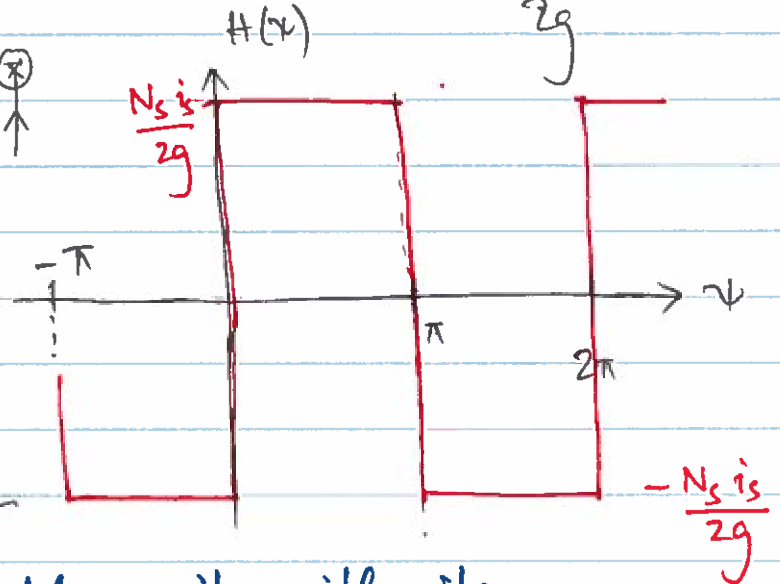
$$\Rightarrow \boxed{H_1 = \frac{N_s i_s}{2g}, \quad H_2 = \frac{-N_s i_s}{2g}.}$$

- Use superposition if you have multiple coils.

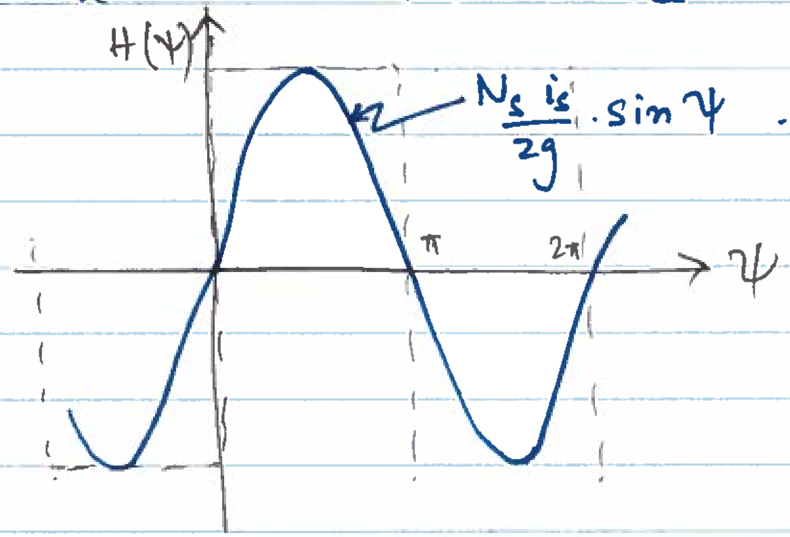
Sinusoidal approximation.

$$H(\psi) = \begin{cases} \dfrac{N_s i_s}{2g}, & 0 < \psi < \pi \\ -\dfrac{N_s i_s}{2g}, & \pi < \psi < 2\pi \end{cases}$$





Replace it with its dominant "Fourier" mode.



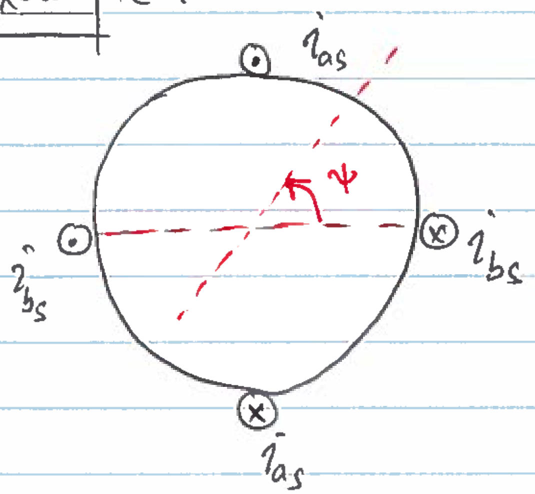
Q. If $i_s = I_s \cos\omega_s t$, or $i_s = I_s \sin\omega_s t$,

$H(\psi)$ is a pulsating function.

- Position of max. remains const
- Maximum value changes with time.

~~Exercises:~~

Example:



N_s turns in both set of windings.

Q1. Find $H(\psi)$ for $0 \le \psi \le 2\pi$.

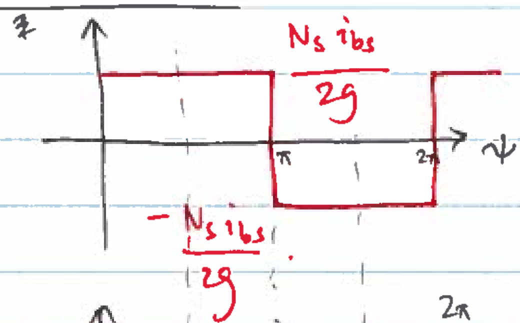
Q2. Find its sinusoidal approximation

Q3. If $i_{as} = I_s \cos \omega_s t$

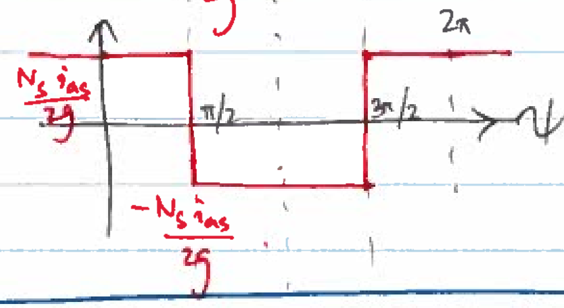
$i_{bs} = I_s \sin \omega_s t$,

find $H(\psi)$ as a function of time. Also, comment whether it's pulsating.

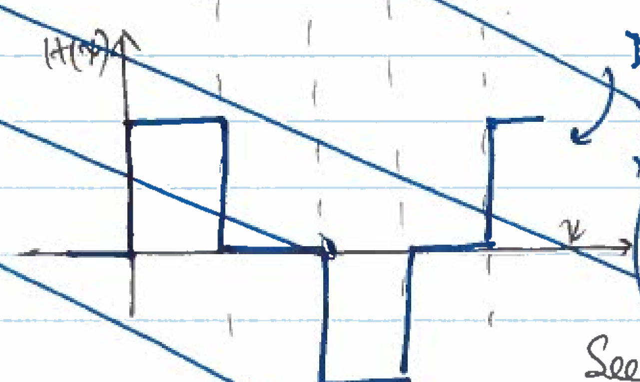
Sol^n: Due to i_{bs}:



Due to i_{as}:



Due to both i_{as} and i_{bs}.



Drawing with the assumption that $i_{as} = i_{bs}$.

Not true otherwise!

See next page

More precisely,

$$H(\psi) = \begin{cases} \frac{N_s}{2g}\left(i_{as} + i_{bs}\right), & 0 < \psi < \pi/2 \\ \frac{N_s}{2g}\left(-i_{as} + i_{bs}\right), & \pi/2 < \psi < \pi \\ \frac{N_s}{2g}\left(-i_{as} - i_{bs}\right), & \pi < \psi < 3\pi/2 \\ \frac{N_s}{2g}\left(i_{as} - i_{bs}\right), & 3\pi/2 < \psi < 2\pi \end{cases}$$

Sinusoidal approx^n: $H(\psi) = \frac{N_s}{2g} i_{bs} \sin\psi + \frac{N_s}{2g} i_{as} \cos\psi$.

When $i_{as} = I_s \cos\omega_s t$, $i_{bs} = I_s \sin\omega_s t$,

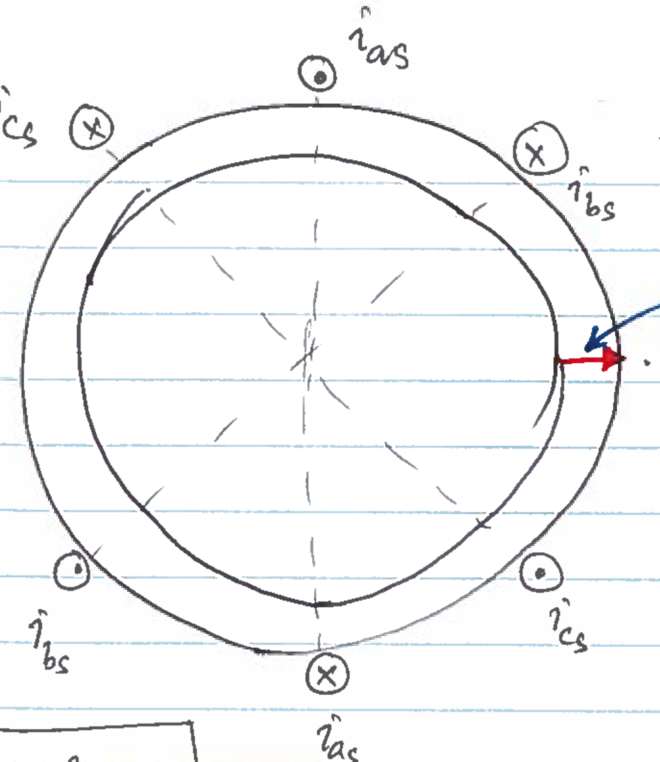
we get, $H(\psi) = \frac{N_s}{2g}\left[\cos\omega_s t \cos\psi + \sin\omega_s t \sin\psi\right]$

$$= \frac{N_s}{2g} \cdot \cos\left(\omega_s t - \psi\right).$$

At time t, max $\{H(\psi)\}$ occurs at ~~...~~ $\psi = \omega_s t$

Not pulsating, but a <u>traveling</u> magnetic field.

or <u>rotating</u>.



Digression (Example):

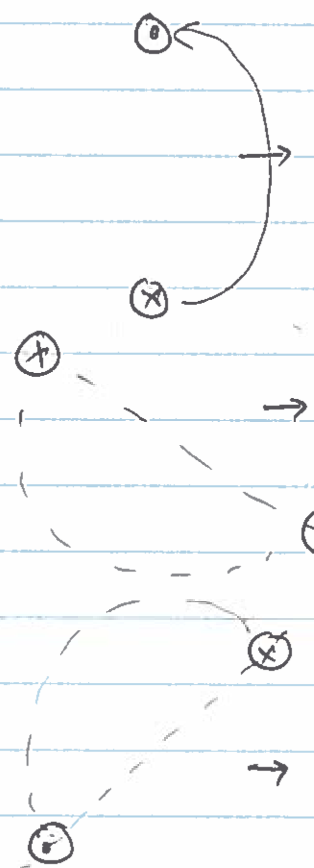
Q. Calculate H here.

Assume all coils have N_s turns.

Thumb-rule:

H gets +ve contribution from ⊗ to ⊙ as you go clockwise.

3 sets of coils.



$\rightarrow \dfrac{N_s\, i_{as}}{2g}$

$\rightarrow -\dfrac{N_s\, i_{cs}}{2g}$

$\rightarrow -\dfrac{N_s\, i_{bs}}{2g}$

Superpose:

$$H = \frac{N_s\, i_{as}}{2g} - \frac{N_s\, i_{cs}}{2g} - \frac{N_s\, i_{bs}}{2g}$$

$$= \frac{N_s}{2g}\left[i_{as} - i_{bs} - i_{cs} \right].$$

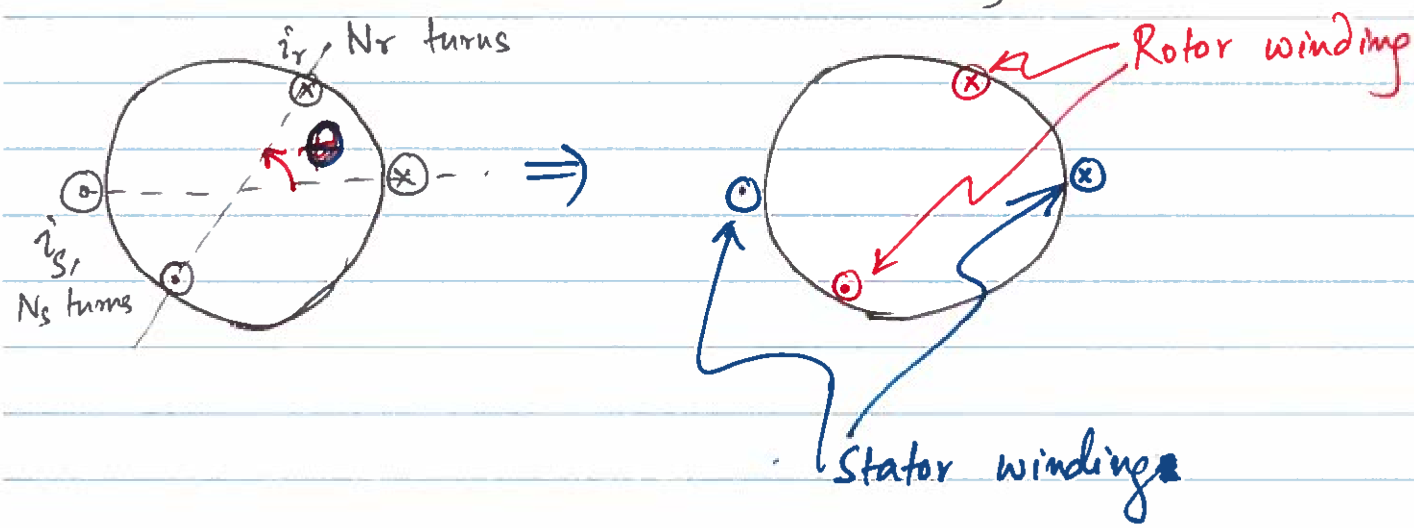
Exercise: Do it for 2ϕ machines.
Remember to use rotor windings as well!

Recall: We were trying to calculate $\lambda = L(\theta)\, i$:

Steps ④ and ⑤.

- Computing $B = \mu_0 H$ is straightforward.
- Computing λ (flux linkage) from B.

Let's do one example. (You can skip the derivation and go to summary in the first reading!)



Q. Find mutual inductance between the two set of windings, with sinusoidal approximation for the magnetic field intensity.

Sol^n: Before doing any work, we can deduce that the answer has the form $(M \cos\theta)$!

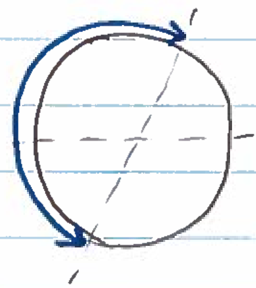
Let's derive it rigorously, and then learn how to get it without doing the work!

$$H(\psi) = \frac{N_s i_s}{2g} \sin(\psi) + \frac{N_r i_r}{2g} \sin(\psi - \theta).$$

How do we compute mutual inductance?

- Zero the current in one coil
- Find flux linkage in the same coil as (mutual inductance) × (current in other coil).

Great!. Let's do it!. Set $i_r = 0$.



To compute flux-linkage through rotor windings, consider a "Curved" surface whose boundary is the blue ~~an~~ line and runs parallel to the ~~ey~~ length of the cylindrical rotor.

$$\Phi = \int_{\text{curved surface}} \vec{B} \cdot d\vec{s} = \int_{\psi=\theta}^{\psi=\theta+\pi} \mu_0 \cdot \frac{N_s i_s}{2g} \sin(\psi)\, d\psi$$

$$= \mu_0 \cdot \frac{N_s i_s}{2g} \cdot \Big[-\cos\psi \Big]_{\theta}^{\theta+\pi}$$

$$= \mu_0 \frac{N_s i_s}{2g} \cdot \Big[\cos\theta - \underbrace{\cos(\theta+\pi)}_{= -\cos\theta} \Big].$$

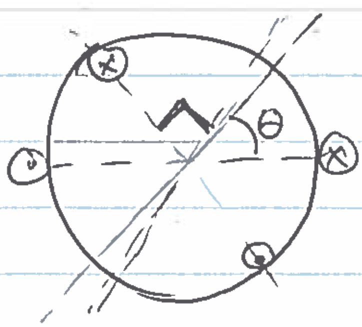
$$\therefore \phi = \frac{\mu_0 N_s i_s}{\cancel{r} g} \cdot \cancel{r} \cos\theta .$$

$$= \frac{\mu_0 N_s i_s}{g} \cos\theta .$$

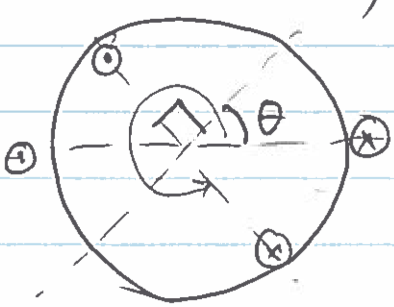
$$\therefore \lambda = N_r \phi = \underbrace{\frac{\mu_0 N_s N_r}{g} \cdot \cos\theta}_{\text{mutual inductance of the form } M\cos\theta .} \cdot \; i_s$$

Summary:

mutual inductance $= M\cos\theta$ where M is a positive constant.



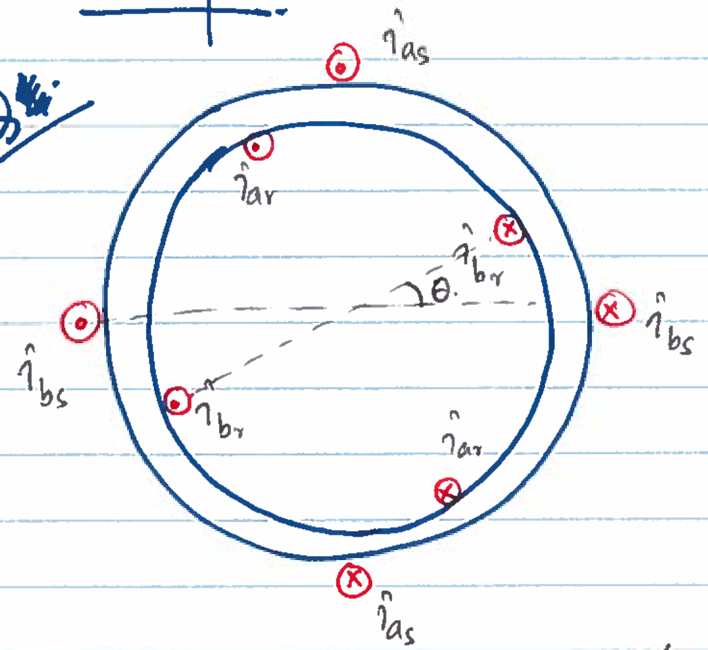
$$\text{mutual inductance} = M\cos\left(\theta + \frac{\pi}{2}\right) = -M\sin\theta$$



$$\text{mutual inductance} = M\cos\left(\theta + \frac{3\pi}{2}\right) = M\sin\theta .$$

Go from ⊗ to other coil's ⊗ anticlockwise to find the angle. Take cos(that angle)!

Example :- (Rather an exercise :)).

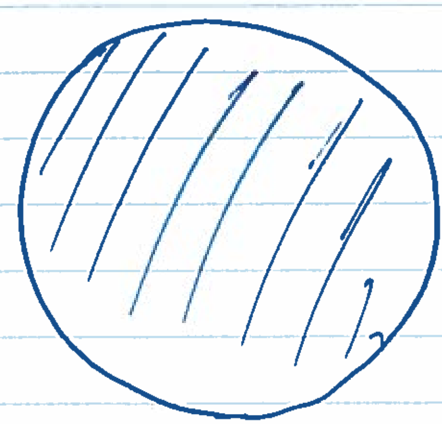


It is tedious, but nothing conceptually difficult. Apply your rules for computing mutual inductances to verify all terms in this $\underline{\lambda} = L(\theta)\underline{i}$ relation.

Verify that

$$\begin{pmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{pmatrix} = \begin{pmatrix} L_s & 0 & M\cos\theta & -M\sin\theta \\ 0 & L_s & M\sin\theta & M\cos\theta \\ M\cos\theta & M\sin\theta & L_r & 0 \\ -M\sin\theta & M\cos\theta & 0 & L_r \end{pmatrix} \begin{pmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{pmatrix}$$

for a 2-ϕ synchronous machine.



Summary of topics under synchronous machines.

Ampere's law → H (radial magnetic field intensity) $\xrightarrow{\text{sinusoidal approx}^n}$ $\underline{\lambda} = L(\theta)\,\underline{i}$

$W_m' = \frac{1}{2}\,\underline{i}^T L(\theta)\,\underline{i}$

↓

$T_e = \frac{\partial W_m'}{\partial \theta}$

↓

Conditions for $\langle T_e \rangle \neq 0$

Some conditions on the frequencies of currents and rotor's angular freq, for steady-state operation.

(In short, "frequency condition")

$v = iR + \frac{d\lambda}{dt}$

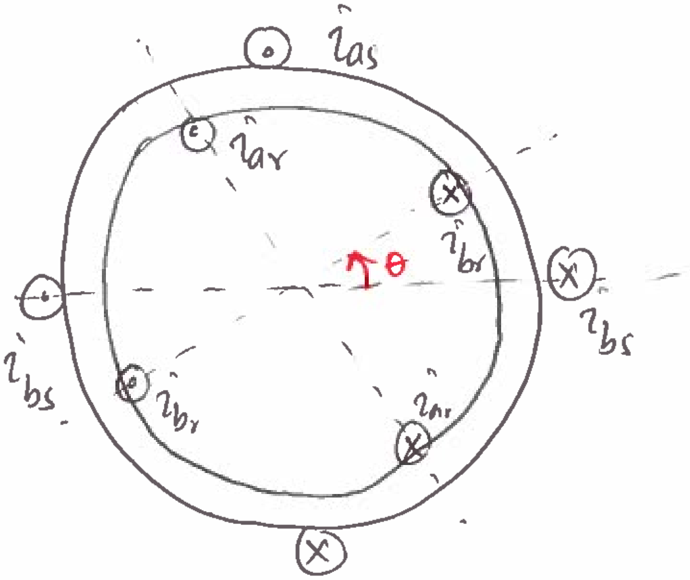
↓

Equivalent circuit.

We know how to do it for 2ϕ sync. machines.

Next step: 3ϕ synchronous machines.

Summary of 2ϕ machines:



① Ampere's law + Sinusoidal approx^n + flux calculation

$$\Downarrow$$

$$\underbrace{\begin{pmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{cs} \\ \lambda_r \end{pmatrix}}_{=\underline{\lambda}} = \underbrace{\begin{pmatrix} L_s & 0 & M\cos\theta & -M\sin\theta \\ 0 & L_s & M\sin\theta & M\cos\theta \\ M\cos\theta & M\sin\theta & L_r & 0 \\ -M\sin\theta & M\cos\theta & 0 & L_r \end{pmatrix}}_{=L(\theta)} \underbrace{\begin{pmatrix} i_{as} \\ i_{bs} \\ i_{cs} \\ i_r \end{pmatrix}}_{\underline{i}}.$$

When $i_{as} = I_s \cos\omega_s t$, $i_{bs} = I_s \sin\omega_s t$,
$i_{ar} = I_r \cos\omega_r t$, $i_{br} = I_r \sin\omega_r t$,
$\theta = \omega_m t + \gamma$

$$\boxed{\omega_m = \omega_s - \omega_r}$$

Frequency condition.

When frequency condition is satisfied,

$$T^e = \langle T^e \rangle = -M I_s I_r \sin\gamma.$$

Mechanical power output $= T^e \omega_m = T^e \omega_s$.

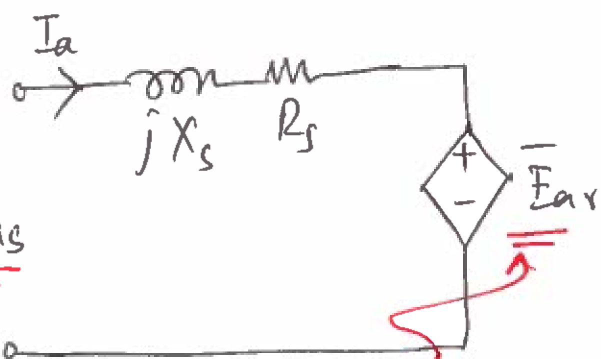
Why? Steady-state $\Rightarrow T^m$ (mech. torque into system) $= -T^e$.

$\therefore$ mech power output $= (-T^m) \cdot \underbrace{\omega_m}_{=\omega_s}$

$= T^e \omega_s$.

What assumptions did we make $\Big\} \Rightarrow T^f = 0$, or no losses in mechanical subsystem.

Equivalent circuit:



called terminal voltage.

called internal voltage.

Fix one of the angles between that of $\bar{V}_{as}$, $\bar{I}_a$, $\bar{E}_{ar}$ to zero.

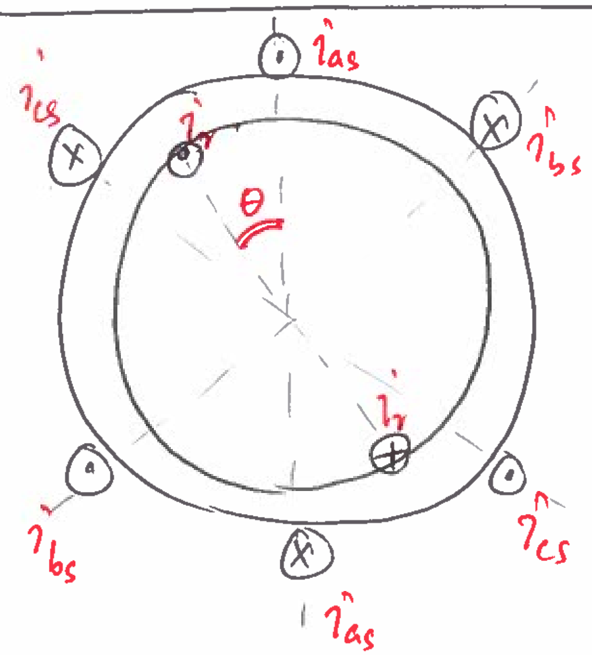
Usually, it's $\bar{V}_{as} = |\bar{V}_{as}| \angle 0^\circ$.

$\measuredangle \bar{I}_a$: can be derived from power factor.

$\measuredangle \bar{E}_{ar}$: δ (call it the torque angle).

angle of $\bar{E}_{ar}$ relative to $\bar{V}_{as}$.

3ϕ- machines (Round rotor)



NOTICE: We only have one rotor coil in this derivation.

$$\begin{pmatrix}\lambda_{as}\\ \lambda_{bs}\\ \lambda_{cs}\\ \lambda_r\end{pmatrix} = \begin{pmatrix} L_0 & -L_0/2 & -L_0/2 & M\cos\theta \\ -L_0/2 & L_0 & -L_0/2 & M\cos(\theta-120^\circ) \\ -L_0/2 & -L_0/2 & L_0 & M\cos(\theta+120^\circ) \\ M\cos\theta & M\cos(\theta-120^\circ) & M\cos(\theta+120^\circ) & L_r \end{pmatrix}\begin{pmatrix} i_{as}\\ i_{bs}\\ i_{cs}\\ i_r\end{pmatrix}$$

You do not need to remember this relation.

But, you should be able to explain why

From last lecture notes.

- Self-inductance of as coil $= L_0$
  $\Rightarrow$ mutual inductance of as, bs coil $= -L_0/2$.
- mutual inductance of as, ar coils $= M\cos\theta$
  $\Rightarrow$ mutual inductance of bs, ar coils $= M\cos(\theta-120^\circ)$.

We shall derive
1. Frequency condition
2. Equivalent circuit.

You do not need to know the derivation, but know the end result!

and under what assumptions are these derived

Assumptions throughout

- $i_{as} = I_s \cos \omega_s t$
- $i_{bs} = I_s \cos(\omega_s t - 120^\circ)$
- $i_{cs} = I_s \cos(\omega_s t + 120^\circ)$
- $i_r = I_r$ [const. and does not vary with time].

- We are interested in steady-state of ^{n}.

$\ddot{\theta} = 0 \Rightarrow \dot{\theta} = \text{const}$

$\Rightarrow \theta = \omega_m t + \gamma$

$$W_m' = \frac{1}{2} \underline{i}^T L(\theta) \underline{i}$$

$$= \frac{1}{2} L_0 i_{as}^2 + \frac{1}{2} L_0 i_{bs}^2 + \frac{1}{2} L_0 i_{cs}^2 + \frac{1}{2} L_r i_r^2$$

$$- \frac{L_0}{2} \left(i_{as} i_{bs} + i_{bs} i_{cs} + i_{cs} i_{as} \right)$$

$$+ M i_a i_r \cos\theta + M i_b i_r \cos(\theta - 120^\circ)$$
$$+ M i_c i_r \cos(\theta + 120^\circ)$$

Deriving frequency condition

Skip to the result directly. Derivation is given to help you understand the details.

$$\frac{\partial W_m'}{\partial \theta} = -M \overset{I_r}{i_r}\left[\underset{I_s \cos \omega_s t}{i_a} \sin \overset{\omega_m t + \gamma}{\theta} + \underset{I_s \cos(\omega_s t - 120^\circ)}{i_b} \sin(\theta - 120^\circ) + \underset{I_s \cos(\omega_s t + 120^\circ)}{i_c} \sin(\theta + 120^\circ)\right].$$

$$= -M I_s I_r \Big[\sin(\omega_m t + \gamma) \cos \omega_s t + \sin(\omega_m t + \gamma - 120^\circ) \cos(\omega_s t - 120^\circ) + \sin(\omega_m t + \gamma + 120^\circ) \cos(\omega_s t + 120^\circ) \Big]$$

Recall : $\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$.

$$\therefore T^e = \frac{\partial W_m'}{\partial \theta} = -\frac{M I_s I_r}{2}\Big[\sin(\omega_m t + \gamma + \omega_s t) + \sin(\omega_m t - \omega_s t + \gamma) + \sin(\omega_m t + \gamma + \omega_s t - 240^\circ) + \sin(\quad) + \sin(\omega_m t + \gamma + \omega_s t + 240^\circ) + \sin(\quad) \Big]$$

same.

this part looks like

$\sin x + \sin(x - 240^\circ) + \sin(x + 240^\circ)$.

That equals 0! Easiest way to see it is through complex nos.

$\sin x = \mathrm{Im}\{e^{jx}\}$.

$\therefore$ above $\exp^n = \mathrm{Im}\Big\{\underbrace{e^{jx} + e^{j(x-240^\circ)} + e^{j(x+240^\circ)}}_{\text{draw these vectors!}}\Big\}$

these sum to zero.

$$T^e = -\frac{M I_s I_r}{2}\left[3\sin\left(\omega_m t - \omega_s t + \gamma\right)\right].$$

$$\boxed{\langle T^e \rangle = \ ? \qquad \text{Same trick as in } 2\phi.}$$

$$\langle T^e \rangle = \begin{cases} 0 & \text{if } \omega_m \neq \omega_s \\ -\dfrac{3 M I_s I_r}{2}\sin\gamma. & \text{if } \omega_m = \omega_s. \end{cases}$$

Result (very important!)

Frequency condition: $\omega_m = \omega_s$.

$$T^e = \langle T^e \rangle = -\frac{3 M I_s I_r}{2}\sin\gamma.$$

Equivalent ckt (steady-state, satisfying frequency condition)

Skip to result directly.
Derivation is only for completeness.

$$v_{as} = i_{as}R_s + \frac{d\lambda_{as}}{dt}$$

$$= \underbrace{i_{as}}_{\hookrightarrow I_s\cos\omega_s t} R_s + \frac{d}{dt}\left[L_0 \overset{I_s\cos\omega_s t}{i_{as}} - \frac{L_0}{2}\overset{I_s\cos(\omega_s t-120^\circ)}{i_{bs}} - \frac{L_0}{2}\overset{I_s\cos(\omega_s t+120^\circ)}{i_{cs}} + M\cos\underbrace{\theta}_{\omega_m t+\gamma}\ \overset{I_r}{i_r}\right]$$

$\omega_m = \omega_s$.

$$v_{as} = I_s \cos w_s t \;\; R_s$$

$$+ \; L_o I_s \left[-w_s \sin w_s t + \frac{w_s}{2} \sin(w_s t - 120^\circ) + \frac{w_s}{2} \sin(w_s t + 120^\circ) \right]$$

$$- M I_r w_s \sin(w_s t + \gamma).$$

$$= I_s R_s \cos w_s t - 3\frac{w_s L_o I_s}{2} \sin w_s t - M I_r w_s \sin(w_s t + \gamma)$$

$$+ \frac{w_s}{2} \Big[\underbrace{\sin w_s t + \sin(w_s t - 120^\circ) + \sin(w_s t + 120^\circ)}_{=0.\ \text{Again, to see this, use the fact } \sin x = \mathrm{Im}\{e^{jx}\}.} \Big]$$

$$= I_s R_s \cos w_s t - \frac{3}{2} w_s L_o I_s \sin w_s t - M I_r w_s \sin(w_s t + \gamma)$$

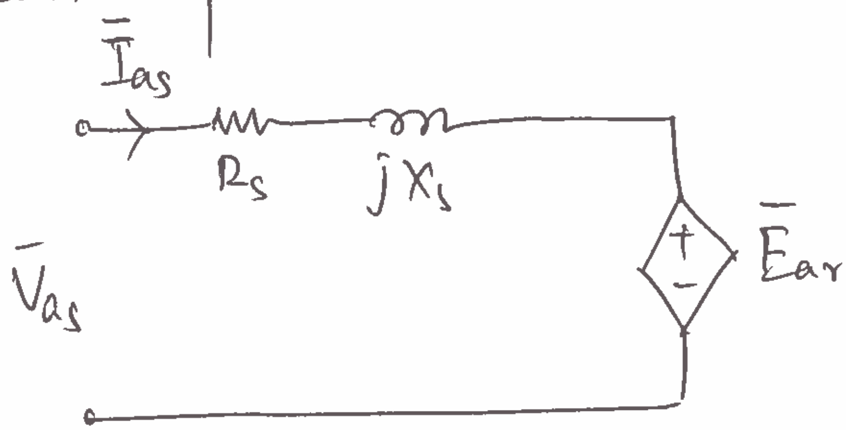
$$= \mathrm{Re}\left\{ \sqrt{2} \left(R_s . \underbrace{\left[\frac{I_s}{\sqrt{2}} \angle 0\right]}_{= \bar{I}_{as}} + \underbrace{\left[j w_s \frac{3 L_o}{2} \right]}_{= j X_s} \underbrace{\left[\frac{I_s}{\sqrt{2}} \angle 0\right]}_{= \bar{I}_{as}} + \underbrace{\left[\frac{M I_r w_s}{\sqrt{2}} \angle \gamma + \frac{\pi}{2} \right]}_{= \bar{E}_{ar}} \right) e^{j w_s t} \right\}.$$

$$\therefore \quad v_{as} = \mathrm{Re}\left\{\sqrt{2}\left(R_s \bar{I}_{as} + jX_s \bar{I}_{as} + \bar{E}_{ar}\right) e^{j\omega_s t}\right\}$$

Then, the phasor $\bar{V}_{as}$ associated with v_{as} satisfies ξ:

$$\bar{V}_{as} = (R_s + jX_s) \cdot \bar{I}_{as} + \bar{E}_{ar}$$

Circuit representation



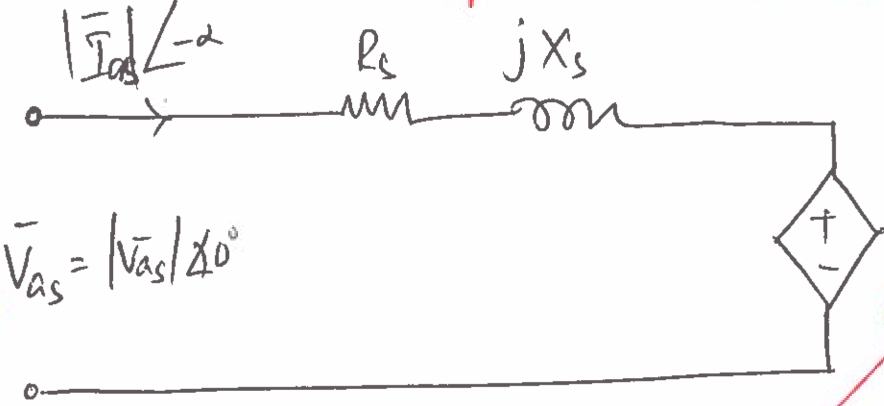
- In this representation, which phasor among $\bar{V}_{as}$, $\bar{I}_{as}$, $\bar{E}_{ar}$ has zero angle? Ans. $\bar{I}_{as}$.
- ~~Rescale~~ Change reference such that $\measuredangle \bar{V}_{as} = 0$.

More precisely, if $\measuredangle \bar{V}_{as} = \alpha$, $\measuredangle \bar{I}_{as} = 0$, $\measuredangle \bar{E}_{ar} = \gamma + \frac{\pi}{2}$.

renormalized values $\left\{ \measuredangle \bar{V}_{as} = 0, \; \measuredangle \bar{I}_{as} = -\alpha, \; \measuredangle \bar{E}_{ar} = \underbrace{\gamma + \frac{\pi}{2} - \alpha}_{} \right.$

why? $\hookrightarrow$ sometimes called <u>torque angle</u> $:= \delta$

Final circuit equivalent (Very important.)



$\bar{V}_{as} = |\bar{V}_{as}|\angle 0°$

terminal voltage

$$\bar{E}_{ar} = \frac{M I_r \omega_s}{\sqrt{2}} \angle \gamma + \frac{\pi}{2} - \alpha$$

$$= \frac{M I_r \omega_s}{\sqrt{2}} \angle \delta .$$

internal voltage.

Next topic:

Using the equivalent circuit.

$S_{3\phi}^{in}$ = Complex power "into" the machine across all three phases

$$= \cancel{3\phi}\ 3\, \underbrace{S_{\phi}^{in}}_{\text{per phase}}$$

$$= 3 . \left[\bar{V}_{as} \bar{I}_{as}^{*} \right].$$

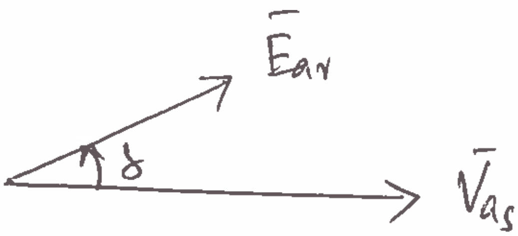
Common assumption: $R_s = 0$, i.e., stator resistance is negligible.

$$\therefore S_{3\phi}^{in} = 3 . \left[\bar{V}_{as} \left(\frac{\bar{V}_{as} - \bar{E}_{ar}}{jX_s} \right)^{*} \right].$$

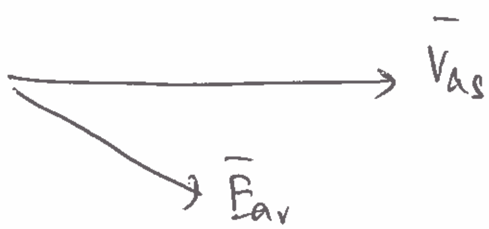
In short, whether the machine is a generator or a motor, is completely encoded in δ.

What is δ again? $\measuredangle$ difference between $\bar{E}_{ar}$ and $\bar{V}_{as}$. .

Phasor-diagram

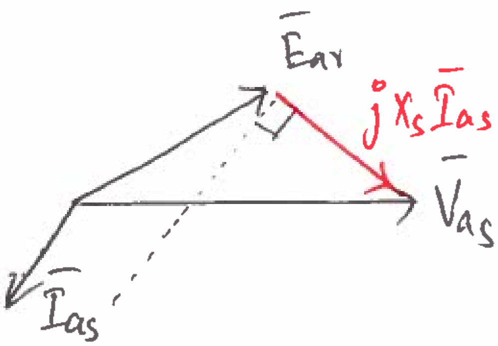


$\Rightarrow$ Generator



$\Rightarrow$ Motor.

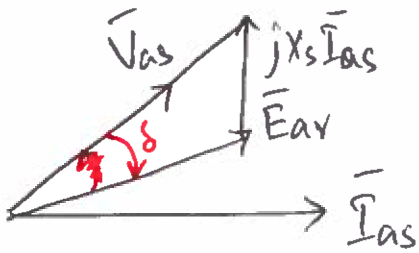
Let's complete the above phasor diagrams! Draw $\bar{I}_{as}$ and $jX_s\bar{I}_{as}$



Recall: $\bar{V}_{as} = \bar{E}_{ar} + jX_s\bar{I}_{as}$

"j" rotates complex no. by $\frac{\pi}{2}$, in the counter-clockwise direction.

Exercise :

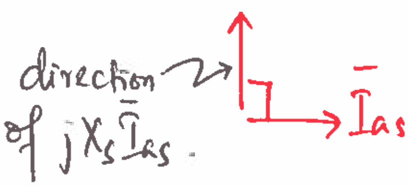


Motor or a generator ?

Ans. All that matters is the angle between $\bar{V}_{as}$ and $\bar{E}_{ar}$.

It's a motor. Verify! .

- Salient point :



$$Q_{3\phi}^{in} = \mathrm{Im}\left\{ S_{3\phi}^{in} \right\}.$$

$$= \frac{3\,|\bar{V}_{as}|}{X_s}\left(|\bar{V}_{as}| - |\bar{E}_{ar}|\cos\delta \right).$$

if $|\bar{V}_{as}| > |\bar{E}_{ar}|\cos\delta$

$\Downarrow$

$Q_{3\phi}^{in} > 0$

Called "UNDER-EXCITED"

if $|\bar{V}_{as}| < |\bar{E}_{ar}|\cos\delta$

$\Downarrow$

$Q_{3\phi}^{in} < 0$

Called "OVER-EXCITED"

EXCITE ? Who excites whom? Recall $|\bar{E}_{ar}| \propto I_r$.

$\therefore$ Rotor current excites $|\bar{E}_{ar}|$.

$$\therefore\; S_{3\phi}^{in} = 3\cdot\left[\frac{|\bar{V}_{as}|^2}{-jX_s} - \frac{\bar{V}_{as}\,\bar{E}_{ar}^{*}}{-jX_s}\right]$$

$|\bar{V}_{as}|\angle 0^\circ$

$|\bar{E}_{ar}|\angle\delta$

$(\cos\delta - j\sin\delta)$

$$= 3\left[j\,\frac{|\bar{V}_{as}|^2}{X_s} - j\,\frac{|\bar{V}_{as}|\cdot|\bar{E}_{ar}|\angle{-\delta}}{X_s}\right].$$

$$= 3\cdot\left[\,j\left(\frac{|\bar{V}_{as}|^2}{X_s} - \frac{|\bar{V}_{as}|\,|\bar{E}_{ar}|}{X_s}\cos\delta\right) - \frac{|\bar{V}_{as}|\,|\bar{E}_{ar}|\sin\delta}{X_s}\right]$$

$$\therefore\; \boxed{P_{3\phi}^{in} = -\,\frac{3\,|\bar{V}_{as}|\,|\bar{E}_{ar}|\cdot\sin\delta}{X_s}}$$

Imp.

Very important.

$$\delta \in [0, \pi] \Rightarrow P_{3\phi}^{in} < 0.$$

$\Downarrow$

~~machine converts electri~~

machine is "pushing out" electrical power.

$\Rightarrow$ GENERATOR.

$$\delta \in [-\pi, 0] \Rightarrow P_{3\phi}^{in} > 0$$

$\Downarrow$

machine is absorbing electrical power ..... converts to mech. power

$\Downarrow$

MOTOR.

Next class:-

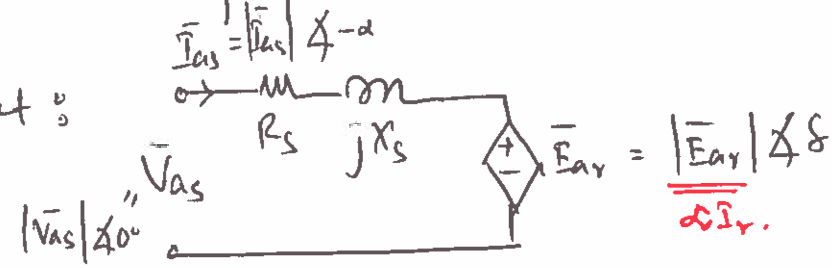
- Do an example of using circuit equivalents of synchronous machines
  → implicitly show you how to model <u>losses</u> in synch machines.

- Start induction machines.

Things to study in synch machines.

$2\phi, 3\phi \longrightarrow$ • $T^e = \langle T^e \rangle$ for sinusoidal currents under freq. cond^n: $\omega_m = \omega_s - \omega_r$.

• Equivalent ckt:



• $P^{in}_{3\phi} = -\dfrac{3|\bar{V}_{as}||\bar{E}_{ar}|}{X_s} \sin \underline{\delta}$

torque angle $\begin{cases} \delta < 0 \Rightarrow \text{~~motor~~ generator} \\ \delta > 0 \Rightarrow \text{motor.} \end{cases}$

• $Q^{in}_{3\phi} = \dfrac{3|\bar{V}_{as}|}{X_s} \left(\underbrace{|\bar{V}_{as}| - |\bar{E}_{ar}| \cos \delta}_{\substack{>0 \,:\, \text{underexcited} \\ <0 \,:\, \text{overexcited.}}} \right)$.