ECE 330
POWER CIRCUITS AND ELECTROMECHANICS

LECTURE 16

STABILITY OF ELECTROMECHANICAL SYSTEMS

Acknowledgment-These handouts and lecture notes given in class are based on material from Prof. Peter Sauer's ECE 330 lecture notes. Some slides are taken from Ali Bazi's presentations

Disclaimer- These handouts only provide highlights and should not be used to replace the course textbook.

NUMERICAL INTEGRATION

There are two kinds of methods: a) explicit and b) implicit.

Euler's method is an explicit one. Euler's method is easier to implement for small systems.

For large systems, the implicit method is better because it is numerically stable.

EXPLICIT METHOD(EULER'S METHOD)

The equation is $\underline{x} = \underline{f}(\underline{x}, \underline{u}),\ \underline{x}(0) = \underline{x}_0$

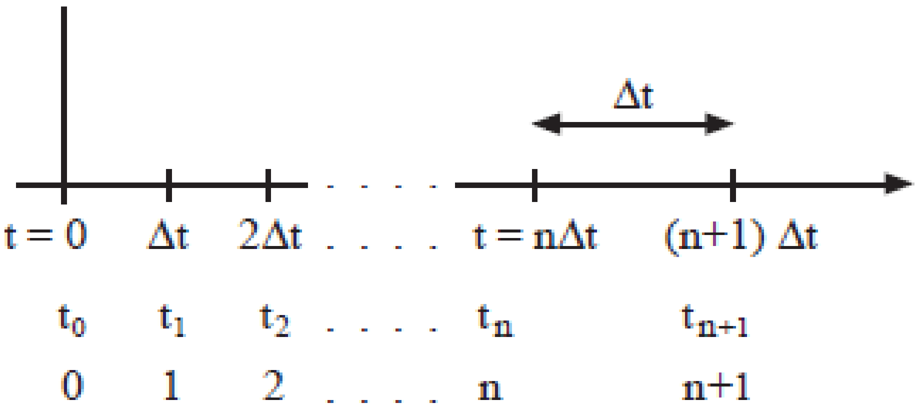
$\underline{x}, \underline{f}$ and u are vectors. We split the time interval into equally spaced time instants $t_0, t_1, t_2, \ldots, t_n, t_{n+1}, \ldots$

From t_n *to* t_{n+1}

$$\int_{t_n}^{t_{n+1}} \underline{x}(t)\,dt = \int_{t_n}^{t_{n+1}} \underline{f}(\underline{x}, \underline{u})\,dt$$

$$\underline{x}(t_{n+1}) - \underline{x}(t_n) = (t_{n+1} - t_n)\ \underline{f}(\underline{x}(t_n),\ \underline{u}(t_n))$$

EXPLICIT METHOD (EULER'S METHOD)



$$\underline{x}(t_{n+1}) = \underline{x}(t_n) + \Delta t\ [\ \underline{f}(\underline{x}(t_n), \underline{u}(t_n))]$$

Where $\Delta t = t_{n+1} - t_n = constant, n = 0,1,2$

Δt is the integration step size

$\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}, \underline{x}^{(n+1)}$ are known from the previous time instant

APPLICATION FOR HIGHER-ORDER SYSTEMS

The same procedure is applicable.

Let
$$\dot{x}_1 = f_1(x_1, x_2, x_3, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, x_3, t)$$
$$\dot{x}_3 = f_3(x_1, x_2, x_3, t)$$

Application of Euler's method gives
$$x_1^{(n+1)} = x_1^{(n)} + \Delta t\,(f_1(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, t_n))$$
$$x_2^{(n+1)} = x_2^{(n)} + \Delta t\,(f_2(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, t_n))$$
$$x_3^{(n+1)} = x_3^{(n)} + \Delta t\,(f_3(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, t_n))$$

Superscript notation is used to denote the time instant

$(n) \Rightarrow n\Delta t = t_n$, $(n+1) \Rightarrow (n+1)\Delta t = t_{n+1}$

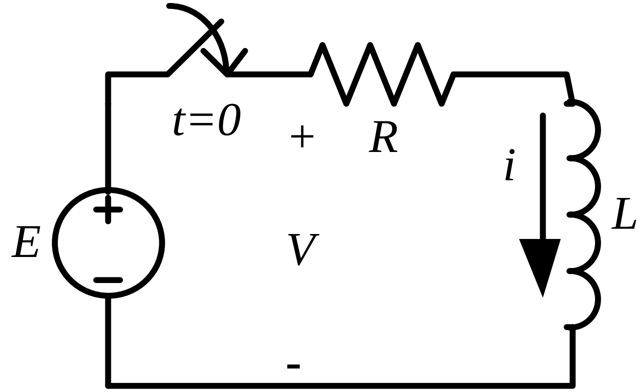
LINEARIZATION

- When we write the equations on the electrical and mechanical side, we have complete dynamic description of the system.
- Then we put them in the state space form as a set of first-order equations for analysis $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$
- Setting $\dot{\underline{x}} = 0$, we get the algebraic equations $0 = \underline{f}(\underline{x}, \hat{\underline{u}})$, which may have several solutions $x_1^e, x_2^e, ...$ called static equilibrium solutions.

LINEARIZATION

- By integrating $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$, $x(0) = x_0$ we can obtain the time domain response.
- a linearized analysis is helpful to determine if the equilibrium point is stable or not by merely computing the eigenvalues of a matrix.
- For large disturbances, a direct method using energy functions can in some cases be used to assess stability without time-domain simulations

LINEAR CIRCUIT MODELING
FIRST-ORDER EXAMPLE



- The circuit differential equation is: $v = i\,R + L\dfrac{di}{dt}$
- State space form $\dfrac{di}{dt} = -\dfrac{R}{L}i + \dfrac{1}{L}v$

$$i(0) = 0,\;\; v(0) = 0 \quad for\;\; t < 0, \quad v = E \quad for\;\; t \geq 0$$

Solution: $i(t) = a e^{\lambda t} + i_{ss}$

FIRST-ORDER EXAMPLE

- DC steady state (equilibrium):

$$0 = -\frac{R}{L}i + \frac{1}{L}E \quad , \quad i_{ss} = \frac{E}{R}$$

- How to solve for λ (eigenvalue)?

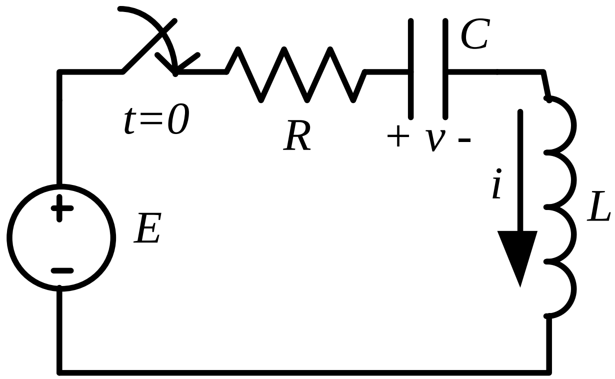
$$|A - \lambda I| = |\lambda I - A| = |\lambda + \frac{R}{L}| = \lambda + \frac{R}{L} \Rightarrow \lambda = -\frac{R}{L}$$

- How do we solve for a? By using initial condition.

$$i(0) = 0 = a + \frac{E}{R} \Rightarrow a = -\frac{E}{R}$$

$$\therefore i(t) = -\frac{E}{R}e^{-\frac{R}{L}t} + \frac{E}{R}$$

SECOND-ORDER EXAMPLE



- The circuit differential equations are:

$$E = i\,R + L\frac{di}{dt} + v \quad , \qquad i = \frac{dv}{dt}$$

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}v + \frac{1}{L}E$$

$$\frac{dv}{dt} = \frac{1}{C}i$$

$$i(0) = 0$$

$$v(0) = V_0$$

SECOND-ORDER EXAMPLE

Then,

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} E$$

The solutions become:

$$i(t) = ae^{\lambda_1 t} + be^{\lambda_2 t} + i_{ss}$$

$$v(t) = ce^{\lambda_1 t} + be^{\lambda_2 t} + v_{ss}$$

The system is stable if Re{λ_1, λ_2}<0

SECOND-ORDER EXAMPLE

At equilibrium:

$$0 = -\frac{R}{L} i - \frac{1}{L} v + \frac{1}{L} E, \quad 0 = \frac{1}{C} i$$

$$i_{ss} = 0, v_{ss} = E$$

Finding the eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & \lambda \end{vmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

$$\lambda_{1,2} = \frac{-R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

The eigenvalues are stable since the real part is <0.

SECOND-ORDER EXAMPLE

$$At \quad t = 0: \quad a + b = i(0), \qquad v_0 = c + d + E$$

$$\left.\frac{di}{dt}\right|_0 = -\frac{R}{L} i(0) - \frac{1}{L} v(0) + \frac{1}{L} E = a\lambda_1 + b\lambda_2$$

$$\left.\frac{dv}{dt}\right|_0 = \frac{1}{C} i(0) = c\lambda_1 + d\lambda_2$$

Solve the following equations to find *a*, *b*, *c*, and *d*

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{E}{L} - \dfrac{V_0}{L} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} V_0 - E \\ 0 \end{bmatrix}$$

LINEARIZATION

A nonlinear system can be expressed as:

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u})$$

The power system may be subjected to small disturbances which may lead to oscillations of power in transmission lines. If not properly damped it may lead to cascading outages and major disturbances (faults).

LINEARIZATION

- Linearization is applied around an equilibrium point x^e and an input $\hat{u}$.
- For a scalar x and u, the Taylor series expansion is:

$$\Delta x = x - x^e$$

$$\Delta u = u - \hat{u}$$

$$i.e., \quad x = x^e + \Delta x, \qquad and \qquad u = \hat{u} + \Delta u$$

$$f(x,u) = f(x^e,\hat{u}) + \left.\frac{\partial f}{\partial x}\right|_{x^e}(x - x^e) + \left.\frac{\partial f}{\partial u}\right|_{x^e}(u - \hat{u}) + h.o.t.$$

LINEARIZATION

- Then,

$$\Delta \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x^e} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x^e} \Delta u$$

- For a second-order system:

$$\dot{\underline{x}}_1 = f_1(\underline{x}_1, \underline{x}_2, \underline{u})$$

$$\dot{\underline{x}}_2 = f_2(\underline{x}_1, \underline{x}_2, \underline{u})$$

$$Let \quad \Delta x_1 = x_1 - x_1^e \ , \quad \Delta x_2 = x_2 - x_2^e \ , \quad \Delta u = u - \hat{u} \, .$$

LINEARIZATION

- Linearizing around the equilibrium point, we get:

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_{x^e} & \left.\dfrac{\partial f_1}{\partial x_2}\right|_{x^e} \\ \left.\dfrac{\partial f_2}{\partial x_1}\right|_{x^e} & \left.\dfrac{\partial f_2}{\partial x_2}\right|_{x^e} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial u}\right|_{x^e} \\ \left.\dfrac{\partial f_2}{\partial u}\right|_{x^e} \end{bmatrix} \Delta u$$

- For an n^{th}- order system, a similar approach is followed.

LINEARIZATION

- The eigenvalues of A determine the stability for small disturbances around the equilibrium point, and obtained by solving for $det[A-\lambda I] = |A-\lambda I| = 0$.
- If all the eigenvalues lie in the left-half plane (i.e., real parts < 0), we say the system is stable. If an

eigenvalue lies in right-half plane, then the system is unstable. If a pair of complex eigenvalues lies on the imaginary axis, the system is *marginally stable*.

EXAMPLE 1

- Linearize the following nonlinear system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \sin(x_1) - 0.866 - x_2$$

- $X_2^e = 0$, $x_1^e = \sin^{-1}(0.866) = 60^o, 120^o$, etc. ; and:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos(x_1) & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

- Eq. pt. 1: Unstable   Eq. pt. 2: Stable

$$A = \begin{bmatrix} 0 & 1 \\ \cos(x_1) & -1 \end{bmatrix}, |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ \frac{1}{2} & -\lambda - 1 \end{vmatrix}$$

$$\lambda_1 = 0.366 > 0, \lambda_2 = -1.366 < 0$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{1}{2} & -\lambda - 1 \end{vmatrix}$$

$$\lambda_1 = -0.5 - 0.5j, \lambda_2 = -0.5 + 0.5j$$

EXAMPLE 2

- Linearize the following nonlinear system:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -6\sin x_1 - 4x_2 + 3$$

- Equilibrium points

$$0 = x_2^e$$

$$0 = -6\sin x_1^e + 3$$

$$x_2^e = 0$$

$$x_1^e = \sin^{-1} 0.5$$

$$= \frac{\pi}{6},\ \frac{5\pi}{6}$$

Linearization

$$\Delta\dot{x}_1 = \Delta x_2$$

$$\Delta\dot{x}_2 = (-6\sin x^e)\Delta x_1 - 4\Delta x_2$$

EXAMPLE 2

$$A = \begin{bmatrix} 0 & 1 \\ -6\cos(x^e) & -4 \end{bmatrix}$$

Compute eigenvalues

$$|A - \lambda I| = \begin{vmatrix} \lambda & -1 \\ 6\cos x^e & \lambda + 4 \end{vmatrix} = \lambda^2 + 4\lambda + 6\cos x^e$$

Characteristic equation

$$\lambda^2 + 4\lambda + 6\cos x^e = 0$$

EXAMPLE 2

$$\lambda = \frac{-4}{2} \pm \frac{1}{2}\sqrt{4^2 - 24\cos x^e}$$

Check stability of two equilibrium points:

1- $x_1^e = \frac{\pi}{6}, \quad x_2^e = 0$

$$\lambda = -2 \pm \frac{1}{2}\sqrt{16 - 24\cos\frac{\pi}{6}}$$

$$= -2 \pm j\,\frac{\sqrt{5}}{2} \qquad \textit{stable}$$

2- $x_1^e = \frac{5\pi}{6}, \quad x_2^e = 0$

$$\lambda = -2 \pm \frac{1}{2}\sqrt{16 - 24\cos\frac{5\pi}{6}}$$

$$= -2 \pm j\,\frac{\sqrt{38}}{2} \qquad \textit{unstable}$$