ECE 330
POWER CIRCUITS AND ELECTROMECHANICS

LECTURE 19
DYNAMICS OF LUMPED MECHANICAL SYSTEMS

Acknowledgment-These handouts and lecture notes given in class are based on material from Prof. Peter Sauer's ECE 330 lecture notes. Some slides are taken from Ali Bazi's presentations

DYNAMICS OF LUMPED MECHANICAL SYSTEMS

So far we have studied the electrical subsystem and its interaction with the lossless magnetic field system, which resulted in the force of electric origin acting on the mechanical system. We next study the dynamics of the mechanical system and finally get the overall dynamic model.

DYNAMICS OF LUMPED MECHANICAL SYSTEMS

Lumped elements of mechanical systems:

- Masses: store kinetic energy(KE)
- Springs: store potential energy(PE)
- Dashpots: the dissipative elements

Elements of electrical systems:

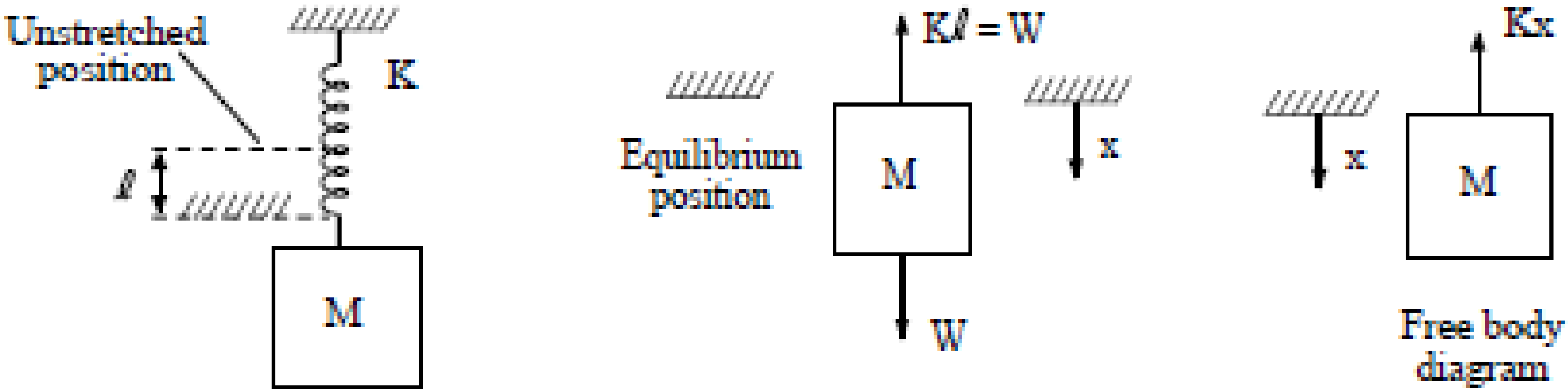
- Capacitance C, inductance L: store energies

- Resistance R: dissipate energy

Instead of KCL and KVL, we have Newton's law to write the equations of motion in a mechanical system.

MASS-SPRING SYSTEM

For a mass $M = \dfrac{W}{g}$ suspended from a spring of stiffness K.



The gravitational force in equilibrium position is

$$W = Mg \text{ is balanced by spring force } K\ell$$

ℓ is the stretching of the spring due to the weight W

MASS-SPRING SYSTEM

Displacement x downward causes a force in the spring to pull the mass upward.

Newton's law: acceleration force in the positive x direction is equal to the algebraic sum of all the forces acting on the mass in the positive x direction.

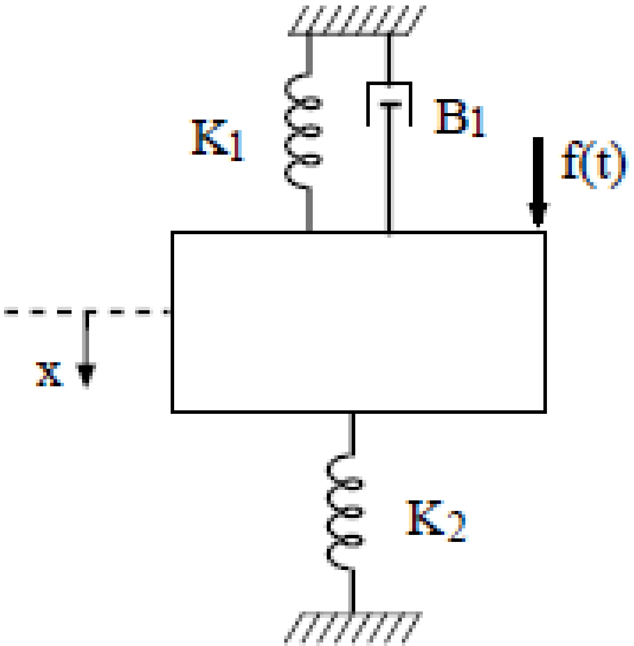
$$M\ddot{x} = -Kx \quad or \quad M\ddot{x} + Kx = 0$$

MASS-SPRING SYSTEM WITH DISSIPATIVE ELEMENT

The same example with dashpot arrangement.

f(t) is the applied force.

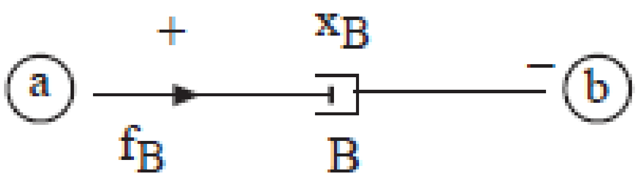
$$f_B = B\frac{dx_B}{dt}$$



Application of Newton's law

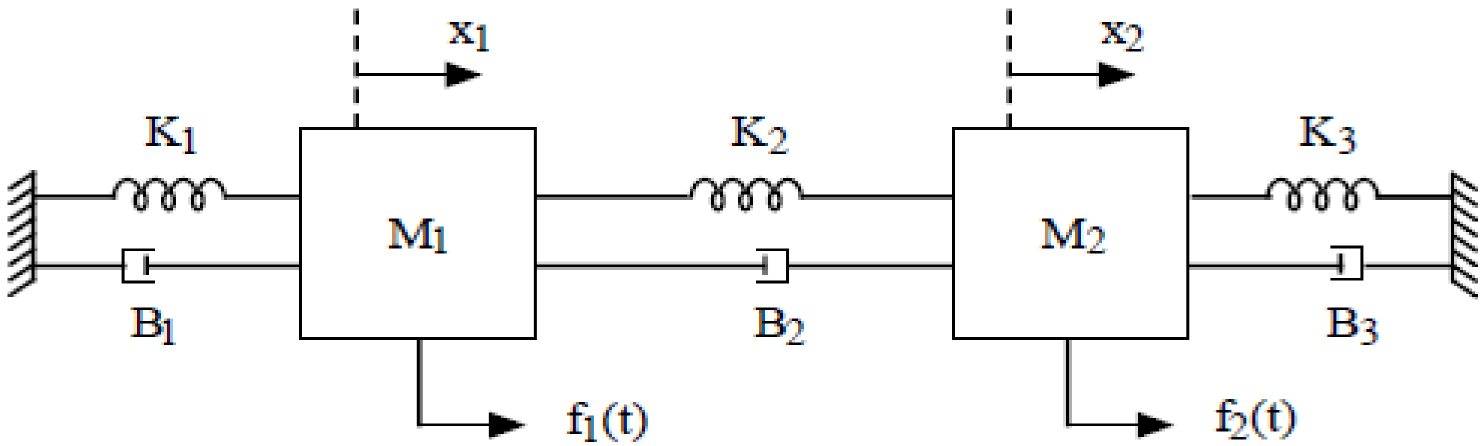
$$M\ddot{x} = f(t) - f_{K1} - f_{K2} - f_B$$

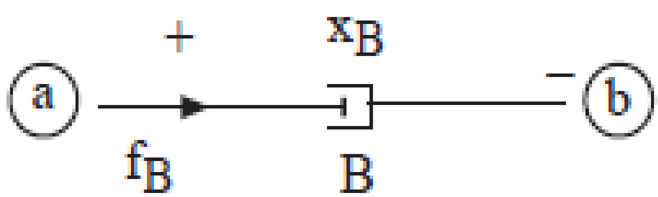
$$M\ddot{x} = f(t) - K_1 x - K_2 x - B\frac{dx}{dt}$$



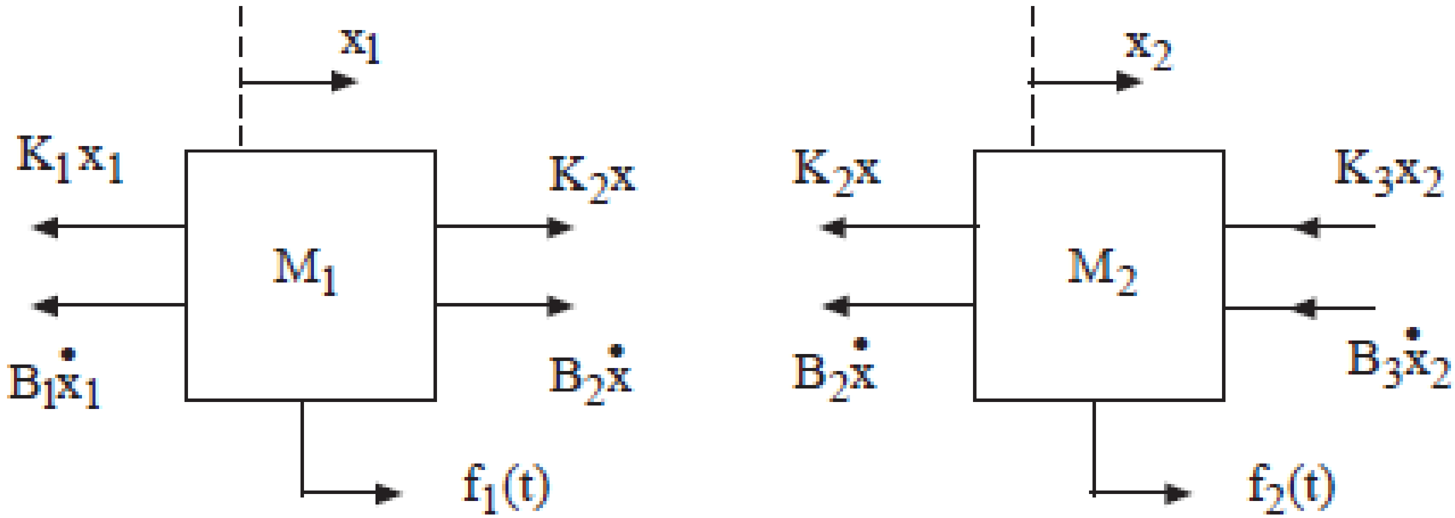
EXAMPLE

The mechanical equations for the two degrees of freedom system. x_1 and x_2 are measured from the equilibrium position





EXAMPLE



The free body diagram

EXAMPLE

Define

$$x_2 - x_1 = x$$

Applying Newton's law to each of the two masses, we get

$$M_1 \ddot{x}_1 = f_1(t) + K_2(x_2 - x_1) + B_2(\dot{x}_2 - \dot{x}_1) - B_1 \dot{x}_1 - K_1 x_1$$

$$M_2 \ddot{x}_2 = f_2(t) - B_2(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) - B_3 \dot{x}_2 - K_3 x_2$$

EXAMPLE

Suppose $f_1(t)$ and $f_2(t)$ are unit step functions of magnitudes

$\hat{F}_1$ and $\hat{F}_2$ respectively.

In steady state the derivatives of x_1 and x_2 are zero.

$$0 = K_1 x_1^{ss} - K_2 (x_1^{ss} - x_2^{ss}) + F_1$$

$$0 = K_3 x_2^{ss} - K_2 (x_1^{ss} - x_2^{ss}) + F_2$$

EXAMPLE

$$\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix} \begin{bmatrix} x_1^{ss} \\ x_2^{ss} \end{bmatrix} = \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix}$$

We solve for x_1^{ss} and x_2^{ss} which will be the new equilibrium positions

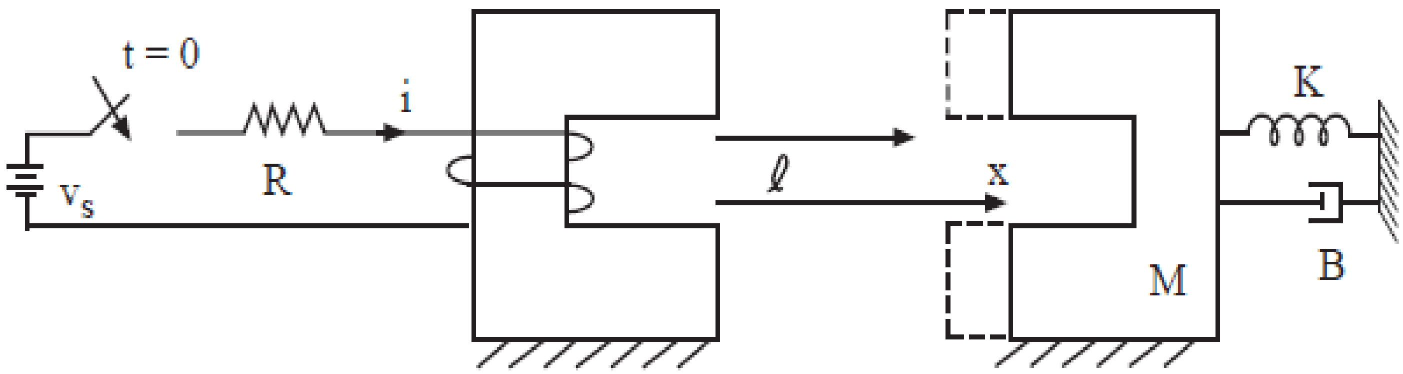
STATE SPACE MODELS

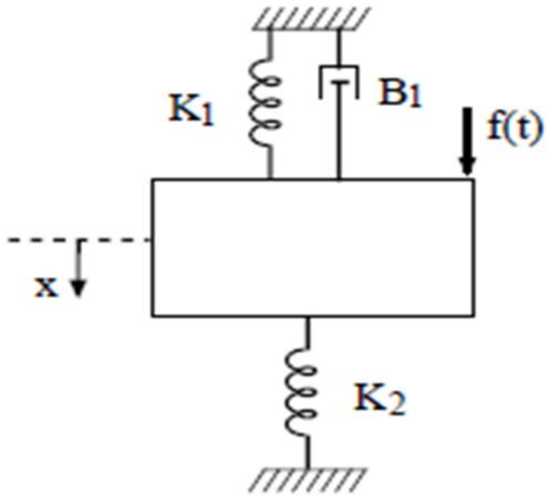
The electrical and mechanical equations are coupled.

State space models are writing the coupled electrical and mechanical equations in the form of a set of first order deferential equations.

EXAMPLE

Write the electrical and mechanical equations of the motion for the system and put them in the state space form.





$$M\ddot{x} = f(t) - K_1 x - K_2 x - B\frac{dx}{dt}$$

EXAMPLE

We have derived the flux linkages in example 4.7:

$$\lambda = \frac{N^2 i}{\left(\frac{\ell_c}{\mu A} + \frac{2x}{\mu_0 A}\right)}$$

Note $\quad \frac{\ell_c}{\mu A} = \Re_c \quad$ and $\quad \frac{2x}{\mu_0 A} = \Re_g(x)$

$$\lambda = \frac{N^2 i}{\Re_c + \Re_g(x)}$$

EXAMPLE

The co-energy is:

$$W'_m = \int_0^i \lambda \, di = \frac{N^2 i^2}{2(R_c + R_g(x))} = \frac{N^2 i^2}{2R(x)}$$

Where, $R(x) = R_c + R_g(x)$

The force of electrical origin: $f^e = \dfrac{\partial W'_m}{\partial x}$

$$f^e = \frac{-N^2 i^2}{\mu_0 A \left(R_c + \dfrac{2x}{\mu_0 A} \right)^2}$$

EXAMPLE

Equations for the electrical side:

$$v_s = i\,R + \frac{d\lambda}{dt}$$

$$v_s = iR + \frac{N^2}{\left(R_c + \frac{2x}{\mu_0 A}\right)}\frac{di}{dt} - \frac{N^2 i}{\left(R_c + \frac{2x}{\mu_0 A}\right)^2}\frac{2}{\mu_0 A}\frac{dx}{dt}$$

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EXAMPLE

Mechanical equation:

Let ℓ be the static equilibrium position of the moving member. This is the same as the upstretched length of the spring with $i = 0$. We assume $\ell > 0$, i.e., the two pieces do not touch each other

EXAMPLE

If we measure the position of the moving member from the equilibrium position, then the mechanical equations have the variable $(x-\ell)$

$$\frac{d^2(x-\ell)}{dt^2}=\frac{d^2x}{dt^2}$$

$$\frac{d(x-\ell)}{dt}=\frac{dx}{dt}$$

$$M\frac{d^2x}{dt^2}+K(x-\ell)+B\frac{dx}{dt}=f^e=\frac{-N^2i^2}{\mu_0A\left(R_c+\frac{2x}{\mu_0A}\right)^2}$$

EXAMPLE

State space model:

A set of three first-order differential equations. Define the state variables as x , $\dfrac{dx}{dt}$ $(denoted\ \ by\ \ \upsilon(velocity))\ and\ \ i$

$$\frac{dx}{dt} = \upsilon\text{..................}(1) \qquad , \qquad \frac{d\upsilon}{dt} = \frac{d^2x}{dt^2}$$

$$\frac{d\upsilon}{dt} = \frac{1}{M}\left[\left(\frac{-N^2 i^2}{\mu_0 A\left(R_c + \dfrac{2x}{\mu_0 A}\right)^2}\right) - K(x-\ell) - B\upsilon\right]\text{..........}(2)$$

EXAMPLE

$$\frac{di}{dt} = \frac{1}{L(x)}\left[-iR + \frac{N^2 i}{\left(R_c + \frac{2x}{\mu_0 A}\right)^2}\frac{2}{\mu_0 A}\upsilon + v_s\right] \ldots\ldots\ldots(3)$$

Where

$$L(x) = \frac{N^2}{\left(R_c + \frac{2x}{\mu_0 A}\right)}$$

EXAMPLE

Equations 1,2,3 are first-order differential equations and are called the state space equations of electromechanical system. The initial conditions are $x(0)=\ell$, $\upsilon(0)=0$, and $i(0)=0$. $v_s(t)$ is a forcing function. Generally, we can also redefine the state variables mathematically as $x=x_1$, $\upsilon=x_2$,

$i=x_3$, and $v_s=u$

EXAMPLE

The state space equations are then of the form

$$\dot{x}_1 = f_1(x_1, x_2, x_3)$$

$$\dot{x}_2 = f_2(x_1, x_2, x_3)$$

$$\dot{x}_3 = f_3(x_1, x_2, x_3, u)$$

EXAMPLE

In the matrix form

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}), \quad \underline{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}, \qquad \underline{u} = u, \quad a\ scalar$$

Where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$\underline{x}(0)$ are the initial conditions. In this case $x_1(0) = \ell$, $x_2(0) = x_3(0) = 0$. Given u(t) and knowing $\underline{x}(0)$, we can integrate to obtain the time domain response.

EQUILIBRIUM POINTS

Consider the equation $\underline{\dot{x}} = f\,(\underline{x}, \underline{u})$, if $\underline{u}$ *is const.* $= \underline{\hat{u}}$,

by setting $\underline{\dot{x}} = 0$ we get algebraic equations $0 = \underline{f}\,(\underline{x}, \underline{\hat{u}})$

This may have several solutions for $\underline{x}$ denoted $\underline{x}_1^e, \underline{x}_2^e$,....

are called *static equilibrium solutions.*

EQUILIBRIUM POINTS

Graphically, setting derivatives equal to zero, $\upsilon^e = 0$,

Since $i^e = \dfrac{v_s}{R}$, x is the only unknown

Plot $-K(x-\ell)$ and $-f^e(i^e, x)$

$$-K(x-\ell) = \frac{N^2 (i^e)^2}{\mu_0 A \left(R_c + \dfrac{2x}{\mu_0 A} \right)^2} = -f^e(i^e, x)$$