ECE330: Power Circuits & Electromechanics

Lecture 14. Co-Energy via line integrals

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Schedule

- ~~Mon 10/7: Voltage of mechanical origin~~
- ~~Wed 10/9: Force of electrical origin~~
- ~~**Friday 10/11: Review + Quiz 5 (Mech. volt.)**~~
- ~~Mon 10/14: Energy via line integrals~~
- Wed 10/16: Co-energy via line integrals
- **Friday 10/18: Review + Quiz 6 (Elec. force)**

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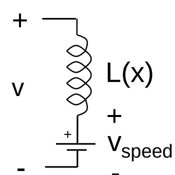
Linear x-varying inductors

Inductance

$$\lambda(i,x) = L(x)i, \quad L(x) = \frac{N^2}{\mathcal{R}(x)}$$

Voltage of mechanical origin

$$v = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + [L'(x)i]\frac{\mathrm{d}x}{\mathrm{d}t}$$



Force of electric origin
(direction of increasing x)

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}] = -\frac{\partial}{\partial x}[\text{Energy}]$$

Energy & Co-energy (This week)

$$\text{Co-energy} = \frac{1}{2}L(x)i^2$$

$$\text{Energy} = \frac{1}{2}L(x)^{-1}\lambda^2$$

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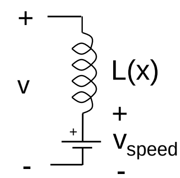
Conservative electromagnetic systems

Flux & current relations

$$\lambda = \lambda(i,x), \quad i = i(\lambda, x)$$

Voltage of mechanical origin

$$v = \underbrace{\frac{\partial\lambda}{\partial i}}_{L(x)}\frac{\mathrm{d}i}{\mathrm{d}t} + \underbrace{\frac{\partial\lambda}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}}_{v_{\text{speed}}}$$



Force of electric origin
(direction of increasing x)

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}] = -\frac{\partial}{\partial x}[\text{Energy}]$$

Energy & Co-energy (Today)

$$\text{Co-energy} = \int_0^i \lambda(i', x)\,\mathrm{d}i'$$

$$\text{Energy} = \int_0^\lambda i(\lambda', x)\,\mathrm{d}\lambda'$$

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Today

- (Review) Line integral definition of energy
- Line integral definition of co-energy
- Example: Forces from co-energy
- Example: Co-energy from flux
- Example: Co-energy from multi-port flux

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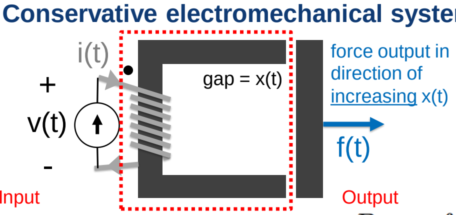
Conservative electromechanical system



Input $P_e = vi$

Output $P_m = f\frac{\mathrm{d}x}{\mathrm{d}t}$

Storage $\frac{\mathrm{d}E}{\mathrm{d}t} = i\frac{\mathrm{d}\lambda}{\mathrm{d}t} - f\frac{\mathrm{d}x}{\mathrm{d}t}$

Energy over Δt

$$\Delta E = i\Delta\lambda - f\Delta x$$

Closed system = <u>Conservative</u>

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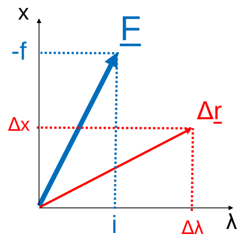
Line integral view of stored energy

$$\begin{aligned}\Delta E &= i\Delta\lambda - f\Delta x \\ &= (i, -f)\cdot(\Delta\lambda, \Delta x) \\ &= \underline{F}\cdot\Delta\underline{r}\end{aligned}$$

Position $\underline{r} = (\lambda, x)$

Force $\underline{F} = (i, -f)$



Flux as spatial dim. Current as force.

etymology for "mmf"

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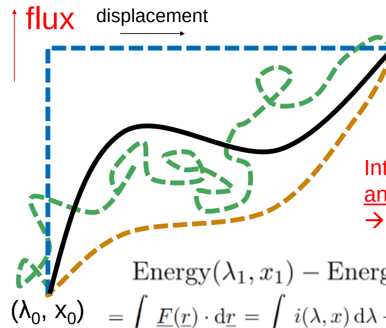
Conservatism implies path independence



Integrate along any path → same energy

$$\mathrm{Energy}(\lambda_1, x_1) - \mathrm{Energy}(\lambda_0, x_0)$$
$$= \int_C \underline{F}(\underline{r})\cdot \mathrm{d}\underline{r} = \int_C i(\lambda, x)\,\mathrm{d}\lambda + \int_C -f(\lambda, x)\,\mathrm{d}x$$

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Line integral definition of energy

$$\mathrm{Energy}(\lambda_1, x_1) - \mathrm{Energy}(\lambda_0, x_0)$$
$$= \int_C \underline{F}(\underline{r})\cdot \mathrm{d}\underline{r} = \underbrace{\int_C i(\lambda, x)\,\mathrm{d}\lambda}_{\text{Energy From Electrical}} + \underbrace{\int_C -f(\lambda, x)\,\mathrm{d}x}_{\text{Energy From Mechanical}}$$

Energy From Electrical

Energy From Mechanical

Path independent (via conservation)

△ Path dependent

$$\mathrm{Energy}(0, x) = 0$$ (Can be any constant)

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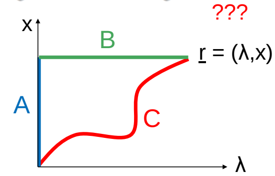
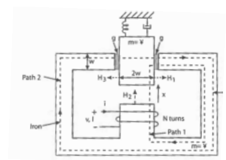
Computing energy from current

$$\mathrm{Energy}(\lambda, x) - \cancel{\mathrm{Energy}(0,0)}^{\,0 \text{ by def}}$$
$$= \int_C i(\lambda', x')\,\mathrm{d}\lambda' + \int_C \underline{-f(\lambda', x')\,\mathrm{d}x'}$$

???



Special integration path avoids the need to know the force



Draw magnetic circuit, compute reluctance & flux, compute flux linkage solve for current

$$i(\lambda, x) = \left(\frac{g + x}{L_0}\right)\lambda$$

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Computing energy from current

$$\mathrm{Energy}(\lambda, x) - \cancel{\mathrm{Energy}(0,0)}^{\,0 \text{ by def}}$$
$$= \int_C i(\lambda', x')\,\mathrm{d}\lambda' + \int_C -f(\lambda', x')\,\mathrm{d}x'$$

no change in flux / no flux no force

$$= \left(\cancel{\int_A i(\lambda', x')\,\mathrm{d}\lambda'}^{\,0} + \cancel{\int_A -f(\lambda', x')\,\mathrm{d}x'}^{\,0}\right)$$

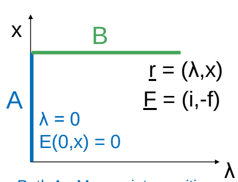
no change in x

$$+ \left(\int_B i(\lambda', x')\,\mathrm{d}\lambda' + \cancel{\int_B -f(\lambda', x')\,\mathrm{d}x'}^{\,0}\right)$$

$$= \int_0^\lambda L(x)^{-1}\lambda'\,\mathrm{d}\lambda' = L(x)^{-1}\left[\frac{(\lambda')^2}{2}\right]_0^\lambda$$

$$\boxed{= \frac{1}{2}L(x)^{-1}\lambda^2}$$

$$i(\lambda, x) = L(x)^{-1}\lambda$$



Path A: Move x into position with no flux

Path B: Charge the inductor with fixed x

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Energy in a conservative magnetic system

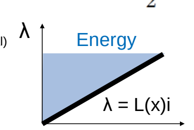
Theorem. If L = ∂λ/∂i > 0, then

$$\mathrm{Energy}(\lambda, x) = \int_0^\lambda i(\lambda', x)\,\mathrm{d}\lambda' \quad [\mathrm{J}]$$

Corollary. If λ = L(x)i and L(x)>0, then

$$\mathrm{Energy}(\lambda, x) = \frac{1}{2}L(x)^{-1}\lambda^2 \quad [\mathrm{J}]$$

Proof. (Graphical)



By definition, energy is the area of the triangle ☐

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Force of electric origin in a conservative magnetic system

Theorem. If energy conserving, then force in the direction of increasing x is

$$f^e = -\frac{\partial}{\partial x}[\text{Energy}]$$

and the current into the dot is

$$i = \frac{\partial}{\partial \lambda}[\text{Energy}]$$

By viewing current as force

Proof. By the gradient theorem

$$\underline{F}(\underline{r}) = (i, -f) = \nabla \text{Energy}(\underline{r})$$
$$\underline{r} = (\lambda, x)$$

$\square$

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Today

- ~~(Review) Line integral definition of energy~~
- Line integral definition of co-energy
- Example: Forces from co-energy
- Example: Co-energy from flux
- Example: Co-energy from multi-port flux

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Main issue with energy computations

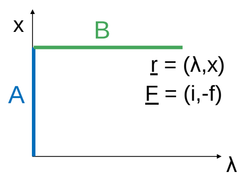
$$\text{Energy}(\lambda,x) - \cancelto{0}{\text{Energy}(0,0)} \quad \text{by def}$$
$$= \int_C i(\lambda',x')\,\mathrm{d}\lambda' + \int_C -f(\lambda',x')\,\mathrm{d}x'$$

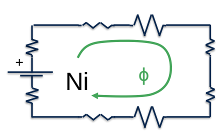
no change in flux — no flux no force

$$= \left(\int_A \cancelto{0}{i(\lambda',x')\,\mathrm{d}\lambda'} + \int_A \cancelto{0}{-f(\lambda',x')\,\mathrm{d}x'}\right)$$

no change in x

$$+ \left(\int_B i(\lambda',x')\,\mathrm{d}\lambda' + \int_B \cancelto{0}{-f(\lambda',x')\,\mathrm{d}x'}\right)$$





Unnatural to express current as function of flux

Can require inverting a nonlinear function

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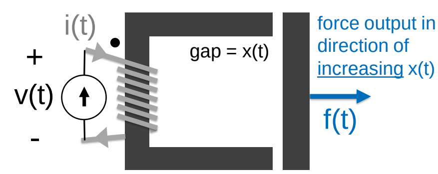
Definition of co-energy



$$\text{Co-energy}(i,x) = \lambda i - \text{Energy}(\lambda, x)$$

$$\frac{\mathrm{d}E_{\text{co}}}{\mathrm{d}t} = \lambda\frac{\mathrm{d}i}{\mathrm{d}t} + i\frac{\mathrm{d}\lambda}{\mathrm{d}t} - \frac{\mathrm{d}E}{\mathrm{d}t}$$

$$= \lambda\frac{\mathrm{d}i}{\mathrm{d}t} + \cancelto{0}{i\frac{\mathrm{d}\lambda}{\mathrm{d}t}} - \left(\cancelto{0}{i\frac{\mathrm{d}\lambda}{\mathrm{d}t}} - f\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \lambda\frac{\mathrm{d}i}{\mathrm{d}t} + f\frac{\mathrm{d}x}{\mathrm{d}t}$$

Swaps the roles of current and flux

positive force

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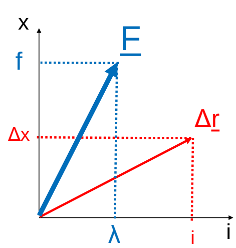
Line integral view of co-energy

$$\frac{\mathrm{d}E_{\text{co}}}{\mathrm{d}t} = \lambda\frac{\mathrm{d}i}{\mathrm{d}t} + f\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\Delta E_{\text{co}} = \lambda\Delta i + f\Delta x$$
$$= (\lambda, f)\cdot(\Delta i, \Delta x)$$
$$= \underline{F}\cdot\Delta\underline{r}$$



Position $\underline{r}$ = (i, x)

Force $\underline{F}$ = (λ, f)

Current as spatial dim. Flux as force.

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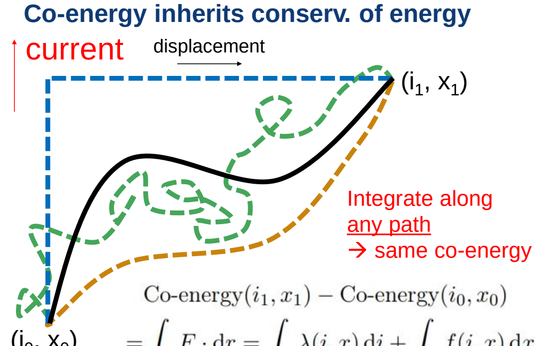
Co-energy inherits conserv. of energy



$$\text{Co-energy}(i_1, x_1) - \text{Co-energy}(i_0, x_0)$$
$$= \int_C \underline{F}\cdot\mathrm{d}\underline{r} = \int_C \lambda(i,x)\,\mathrm{d}i + \int_C f(i,x)\,\mathrm{d}x$$

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Line integral definition of co-energy

$$\text{Co-energy}(i_1, x_1) - \text{Co-energy}(i_0, x_0)$$

$$= \int_C \underline{F} \cdot \mathrm{d}\underline{r} = \int_C \lambda(i,x)\,\mathrm{d}i + \int_C f(i,x)\,\mathrm{d}x$$

Path independent (via conservation)

Path dependent

Not necessarily physical

$$\text{Co-energy}\,(0,x) = 0$$ (Can be any constant)

$$\text{Co-energy}(i,x) = \lambda i - \text{Energy}(\lambda, x)$$

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Computing co-energy from flux

$$\text{Co-energy}(i,x) - \cancel{\text{Co-energy}(0,0)}$$

0 by def

$$= \int_C \lambda(i',x')\,\mathrm{d}i' + \int_C f(i',x')\,\mathrm{d}x'$$

no change in current — 0; no current no force — 0

$$= \left(\int_A \cancel{\lambda(i',x')\,\mathrm{d}i'} + \int_A \cancel{f(i',x')\,\mathrm{d}x'}\right)$$

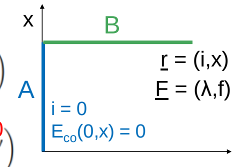
no change in x — 0

$$+ \left(\int_B \lambda(i',x')\,\mathrm{d}i' + \int_B \cancel{f(i',x')\,\mathrm{d}x'}\right)$$

$$= \int_0^i L(x) i'\,\mathrm{d}i' = \left[L(x)\frac{(i')^2}{2}\right]_0^i$$

$$\boxed{= \frac{1}{2} L(x) i^2}$$

$$\lambda(i,x) = L(x) i$$



Path A: Move x into position with no current

Path B: Charge the inductor with fixed x

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Co-energy in a conservative magnetic system

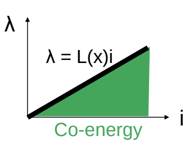
Theorem. If $L = \partial\lambda/\partial i > 0$, then

$$\text{Co-energy}(i,x) = \int_0^i \lambda(i',x)\,\mathrm{d}i' \quad \text{[J]}$$

Corollary. If $\lambda = L(x)i$ and $L(x)>0$, then

$$\text{Co-energy}(i,x) = \frac{1}{2} L(x) i^2 \quad \text{[J]}$$

Proof. (Graphical)



By definition, co-energy is the area of the triangle

□

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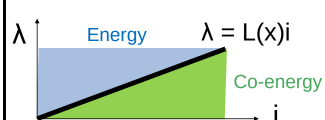
Graphical relationship

$$\text{Energy}(\lambda, x) = \int_0^\lambda i(\lambda', x)\,\mathrm{d}\lambda'$$

$$\text{Co-energy}(i,x) = \int_0^i \lambda(i',x)\,\mathrm{d}i'$$

$$= \lambda i - \text{Energy}(\lambda, x)$$



Energy = co-energy if and only if linear λ-i relation

λ vs i (space-force). not λ vs x (space-space).

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Force of electric origin in a conservative magnetic system

Theorem. If energy conserving, then force in the direction of increasing x is

$$f^e = +\frac{\partial}{\partial x}\,[\text{Co-energy}]$$

and the current into the dot is

$$\lambda = +\frac{\partial}{\partial i}\,[\text{Co-energy}]$$

Again, by viewing flux as force

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Today

- ~~(Review) Line integral definition of energy~~
- ~~Line integral definition of co-energy~~
- Example: Forces from co-energy
- Example: Co-energy from flux
- Example: Co-energy from multi-port flux

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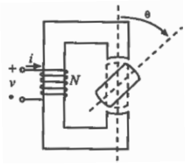
Warm-up: Forces from co-energy



$$W'_m(i,\theta) = \log(1+i)\cos\theta$$

Recall $\lambda = +\dfrac{\partial}{\partial i}[\text{Co-energy}]$

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]$$ Treat θ as x

$\lambda =?$ [At]

A) $+\dfrac{\cos\theta}{1+i}$

~~B)~~ $\cancel{-\dfrac{\cos\theta}{1+i}}$

~~C)~~ $\cancel{+\dfrac{\cos\theta}{1+i} - \log(1+i)\sin\theta}$

~~D)~~ $\cancel{\dfrac{\cos\theta}{1+i} - \log(1+i)\sin\theta}$

$f^e =?$ [N]

~~A)~~ $\cancel{+\log(1+i)\sin\theta}$

B) $-\log(1+i)\sin\theta$

~~C)~~ $\cancel{+\dfrac{\cos\theta}{1+i} - \log(1+i)\sin\theta}$

~~D)~~ $\cancel{\dfrac{\cos\theta}{1+i} - \log(1+i)\sin\theta}$

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Today

- ~~(Review) Line integral definition of energy~~
- ~~Line integral definition of co-energy~~
- ~~Example: Forces from co-energy~~
- Example: Co-energy from flux
- Example: Co-energy from multi-port flux

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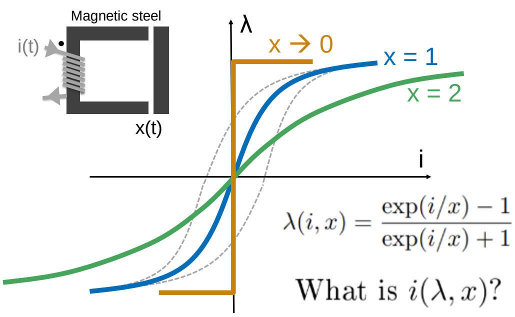
Realistic inductor (without hysteresis)



$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$

What is $i(\lambda, x)$?

How does saturation affect force?

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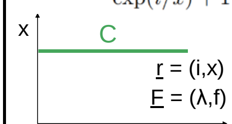
Computing co-energy from flux

$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$



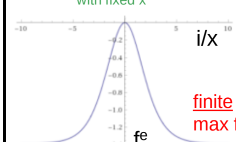
Path C: Charge the inductor with fixed x

$$\text{Co-energy}(i,x) - \cancel{\text{Co-energy}(0,x)}$$ by def 0

$$= \int_C \lambda(i',x')\,\mathrm{d}i' + \cancel{\int_C f(i',x')\,\mathrm{d}x'}$$ 0 no change in x

$$= \int_0^i \frac{\exp(i'/x)-1}{\exp(i'/x)+1}\,\mathrm{d}i'$$

$$\boxed{= 2x\log\left(\frac{\exp(i/x)+1}{2}\right) - i}$$



finite max force

$$f^e = +\frac{\partial}{\partial x}[\text{Co-energy}]$$

$$= \frac{-2i/x}{1+\exp(-i/x)}$$

$$+ 2\log\left(\frac{\exp(i/x)+1}{2}\right)$$

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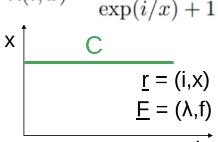
Computing co-energy from flux

$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$



Path C: Charge the inductor with fixed x

$$\text{Co-energy}(i,x) - \cancel{\text{Co-energy}(0,x)}$$ by def 0

$$= \int_C \lambda(i',x')\,\mathrm{d}i' + \cancel{\int_C f(i',x')\,\mathrm{d}x'}$$ 0 no change in x

$$= \int_0^i \frac{\exp(i'/x)-1}{\exp(i'/x)+1}\,\mathrm{d}i'$$

$$\boxed{= 2x\log\left(\frac{\exp(i/x)+1}{2}\right) - i}$$

$$\frac{\partial}{\partial i}[\text{Co-energy}] = 2x \cdot \frac{2}{\exp(i/x)+1} \cdot \frac{\exp(i/x)}{2} \cdot \frac{1}{x} - 1$$

$$= \frac{2\exp(i/x)}{\exp(i/x)+1} - 1 = \lambda(i,x)$$

Partial derivative wrt current recovers flux, as expected.

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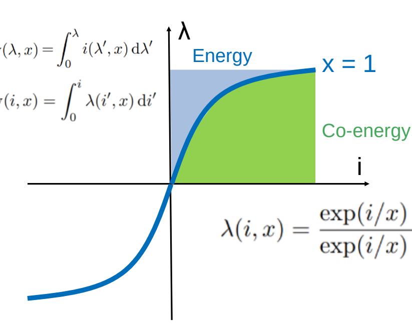
E and co-E in a realistic inductor

$$\text{Energy}(\lambda,x) = \int_0^\lambda i(\lambda',x)\,\mathrm{d}\lambda'$$

$$\text{Co-energy}(i,x) = \int_0^i \lambda(i',x)\,\mathrm{d}i'$$



$$\lambda(i,x) = \frac{\exp(i/x)-1}{\exp(i/x)+1}$$

In general, Energy ≠ Co-energy

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Today

- ~~(Review) Line integral definition of energy~~
- ~~Line integral definition of co-energy~~
- ~~Example: Forces from co-energy~~
- ~~Example: Co-energy from flux~~
- Example: Co-energy from multi-port flux

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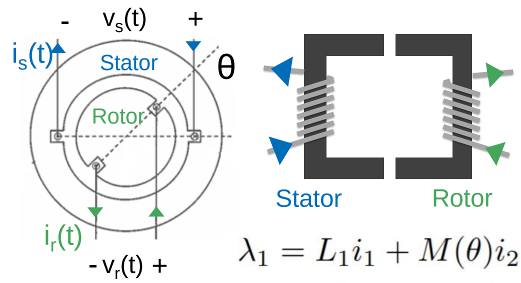
Computing co-energy from multiport flux



$$\lambda_1 = L_1 i_1 + M(\theta) i_2$$
$$\lambda_2 = M(\theta) i_1 + L_2 i_2$$

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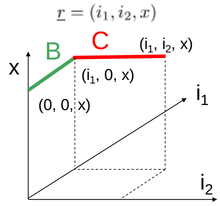
Integration path in two-port system

$\underline{F} = (\lambda_1, \lambda_2, f)$ $\qquad$ $\lambda_1 = L_1 i_1 + M(\theta) i_2$

$\underline{r} = (i_1, i_2, x)$ $\qquad$ $\lambda_2 = M(\theta) i_1 + L_2 i_2$



B: Charge coil 1 with fixed x and de-energized coil 2

C: Charge coil 2 with fixed x and energized coil 1

$$\text{Co-energy}(i_1, i_2, \theta)$$
$$= \underbrace{\int_0^{i_1} \lambda_1(i_1', 0, \theta)\, di_1'}_{\text{Charge coil 1}} + \underbrace{\int_0^{i_2} \lambda_2(i_1, i_2', \theta)\, di_2'}_{\text{Charge coil 2 with coil 1 charged}}$$
$$= \int_0^{i_1} [L_1 i_1' + M(\theta)\, 0]\, di_1'$$
$$+ \int_0^{i_2} [M(\theta) i_1 + L_2 i_2']\, di_2'$$
$$\boxed{= L_1 \frac{i_1^2}{2} + M(\theta) i_1 i_2 + L_2 \frac{i_2^2}{2}}$$
$$= \frac{1}{2} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}^T \begin{bmatrix} L_1 & M(\theta) \\ M(\theta) & L_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

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Co-energy in a conservative magnetic system with multiple ports

Theorem. If $\underline{\underline{L}} = \nabla_{\underline{i}} \underline{\lambda} \succ 0$, then

always fixed

$$\text{Co-energy}(i_1, i_2, i_3, \ldots, x) = \int_0^{i_1} \lambda_1(i_1', 0, 0, \ldots, x)\, di_1'$$
$$+ \int_0^{i_2} \lambda_1(i_1, i_2', 0, \ldots, x)\, di_2'$$
$$+ \int_0^{i_3} \lambda_1(i_1, i_2, i_3', \ldots, x)\, di_3'$$
$$+ \cdots$$

"Charge one coil at a time"

Corollary. If $\underline{\lambda} = \underline{\underline{L}}(x)\underline{i}$ and $\underline{\underline{L}}(x) \succ 0$, then

$$\text{Co-energy}(\underline{i}, x) = \frac{1}{2} \underline{i}^T \underline{\underline{L}}(x) \underline{i} \quad \text{[J]}$$

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Schedule

- ~~Mon 10/7: Voltage of mechanical origin~~
- ~~Wed 10/9: Force of electrical origin~~
- ~~**Friday 10/11: Review + Quiz 5 (Mech. volt.)**~~
- ~~Mon 10/14: Energy via line integrals~~
- ~~Wed 10/16: Co-energy via line integrals~~
- **Friday 10/18: Review + Quiz 6 (Elec. force)**

Do homework 6
For example answers, look at Problems 1 & 2 from previous Exam 2

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