ECE330: Power Circuits & Electromechanics

Lecture 2. Complex Power

Prof. Richard Y. Zhang
Univ. of Illinois at Urbana-Champaign
ryz@illinois.edu

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Schedule

- ~~Mon 8/26: Phasors~~
- Wed 8/28: Complex power
- Fri 8/30: Power factor correction
- Mon 9/2: Labor day (no class)
- Wed 9/4: Three-phase power
- **Fri 9/6: Review + Quiz 1**

Announcements

- iClicker REEF
- Piazza

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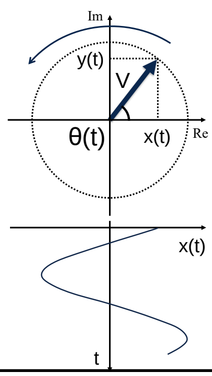
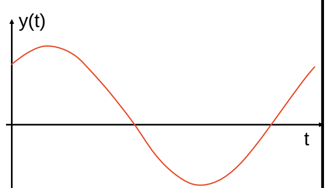
Last lecture. Phasors





$$\theta(t) = \underline{\phi} + \underline{\omega t}$$

Phase — Angular velocity

$$y(t) = V\sin(\omega t + \phi)$$
$$x(t) = V\cos(\omega t + \phi)$$

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Last lecture. Phasors

~~Phasor (peak)~~

~~$\overline{V}_p = V\angle\phi$~~

~~Imaginary axis~~

~~$y(t) = V\sin(\omega t + \phi)$~~

Phasor (rms)

$$\overline{V}_{rms} = \frac{V}{\sqrt{2}}\angle\phi$$

Real axis

$$x(t) = V\cos(\omega t + \phi)$$

Sine vs Cosine: Always assume **cosine-based**

Peak vs RMS: Always assume **RMS**

Amplitude must always be **positive**

Power engineer's convention.

Make sure you memorize this slide

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Today

- Review: Product of complex numbers
- Product of sinusoidal waves
- Complex power and the power triangle
- Exercises

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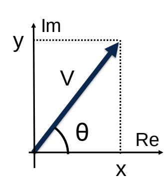
Review: Complex numbers



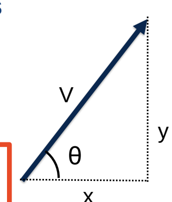
$$x^2 + y^2 = V^2$$
$$\tan\theta = y/x$$

Rectangular

$$x + yj$$
$$= V\angle\theta$$

Polar

$$= Ve^{j\theta}$$



$$x = V\cos\theta$$
$$y = V\sin\theta$$

$$x + jy = (V\cos\theta) + j(V\sin\theta)$$
$$= V(\cos\theta + j\sin\theta) = Ve^{j\theta}$$

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Aside: Euler's Identity

Theorem (Euler). $e^{j\theta} = \cos\theta + j\sin\theta$

Proof.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} + \cdots$$

$$= 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos x + j\sin x \qquad \square$$

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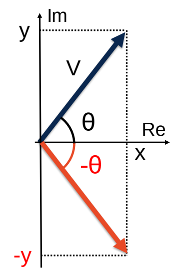
Review: Complex Conjugates



$$x + yj = V\measuredangle\theta = Ve^{j\theta}$$

Star denotes "conjugate"

$$(x + yj)^* = x - yj$$
$$= V\measuredangle(-\theta)$$
$$= Ve^{-j\theta}$$

Reflection across the real line

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Review: Complex multiplication

Rectangular: Complicated & error-prone

$$\overline{V} = V_r + jV_i, \qquad \overline{I} = I_r + jI_i.$$

$$\overline{V}\overline{I}^* = (V_r + jV_i)(I_r + jI_i)^*$$
$$= (V_r + jV_i)(I_r - jI_i)$$
$$= (V_rI_r - V_iI_i) + j(V_iI_r - V_rI_i)$$

Polar: Intuitive & hard to get wrong

$$\overline{V} = Ve^{j\theta}, \qquad \overline{I} = Ie^{j\phi}.$$

$$\overline{V}\overline{I}^* = (Ve^{j\theta})(Ie^{j\phi})^* = (Ve^{j\theta})(Ie^{-j\phi})$$
$$= (VI)(e^{j\theta}e^{-j\phi}) = (VI)e^{j(\theta-\phi)}$$

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Product of Sinusoids

High school trig (remember this?)

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$$

sum of angles — diff of angles

Now consider cosine waves

$$\cos(\omega_1 t)\cdot\cos(\omega_2 t) = +\frac{1}{2}\cos[(\omega_1+\omega_2)t] + \frac{1}{2}\cos[(\omega_1-\omega_2)t]$$

sum of frequencies — diff of frequencies

Coincidence?

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Product of Sinusoids

$$\sin(\omega_1 t)\cdot\sin(\omega_2 t) = -\frac{1}{2}\cos[(\omega_1+\omega_2)t] + \frac{1}{2}\cos[(\omega_1-\omega_2)t]$$

sum of frequencies — diff of frequencies

$$\cos(\omega_1 t)\cdot\sin(\omega_2 t) = +\frac{1}{2}\sin[(\omega_1+\omega_2)t] + \frac{1}{2}\sin[(\omega_1-\omega_2)t]$$

sum of frequencies — diff of frequencies

$$\cos(\omega_1 t)\cdot\cos(\omega_2 t) = +\frac{1}{2}\cos[(\omega_1+\omega_2)t] + \frac{1}{2}\cos[(\omega_1-\omega_2)t]$$

sum of frequencies — diff of frequencies

Look for counterexamples

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Product of Sinusoids: Same Frequency

Again, consider cosine waves

$$\cos(\omega_1 t)\cdot\cos(\omega_2 t) = +\frac{1}{2}\cos[(\omega_1+\omega_2)t] + \frac{1}{2}\cos[(\omega_1-\omega_2)t]$$

Set both frequencies $\omega_1 = \omega_2 = \omega$

$$\cos(\omega t)\cdot\cos(\omega t) = +\frac{1}{2}\cos(2\omega t) \quad + \frac{1}{2}$$

double frequency — DC

Let's see a real-world example

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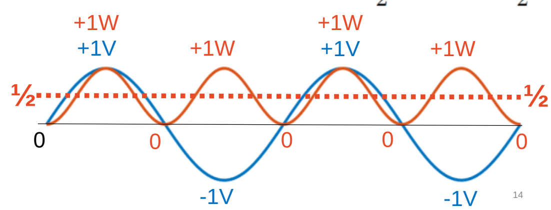
Powering a lightbulb

$$v(t) = \cos(\omega t)$$

$$i(t) = \frac{v(t)}{R} = \cos(\omega t)$$

$$p(t) = v(t)i(t) = \frac{1}{2}\cos(2\omega t) + \frac{1}{2}$$

v(t) = cos(ωt)



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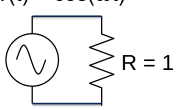
Powering a lightbulb

v(t) = cos(ωt)



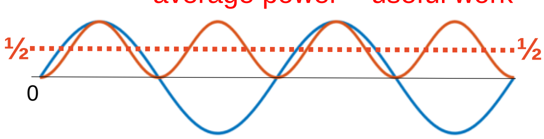
Instantaneous power

$$p(t) = \frac{1}{2}\cos(2\omega t) + \frac{1}{2}$$

Average power

$$\langle p(t)\rangle = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t$$

average power = useful work



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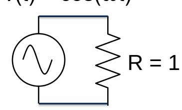
Powering a lightbulb

v(t) = cos(ωt)



Instantaneous power

$$p(t) = \frac{1}{2}\cos(2\omega t) + \frac{1}{2}$$

Average power

$$\langle p(t)\rangle = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t$$

$$\langle p(t)\rangle = \frac{1}{T}\int_0^T \left(\frac{1}{2}\cos(2\omega t)\right)\mathrm{d}t + \frac{1}{T}\int_0^T \left(\frac{1}{2}\right)\mathrm{d}t$$

$$= 0 \qquad\qquad = \frac{1}{2}$$

Only DC component does useful work

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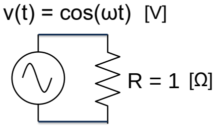
Why RMS?

v(t) = cos(ωt) [V]



$$v(t) = \cos(\omega t)\ \text{[V]}$$

$$i(t) = \frac{v(t)}{R} = \cos(\omega t)\ \text{[A]}$$

$$p(t) = v(t)i(t)\ \text{[W]}$$

$$= \frac{1}{2}\cos(2\omega t) + \frac{1}{2}$$

$$\langle p(t)\rangle = \qquad + \frac{1}{2}\ \text{[W]}$$

1 V_{pk} → 1 Ω → 1 A_{pk} → ½ W ??

Factor of ½ is confusing.

It's also different for square waves, triangle waves, sawtooth waves

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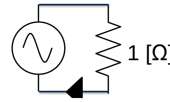
Why RMS? Exact analog to DC.

$\bar{V}_{rms} = 1\angle 0°$ [V]



$\bar{I}_{rms} = 1\angle 0°$ [A]

$$v(t) = \sqrt{2}\cos(\omega t)\ \text{[V]}$$

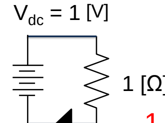
$$i(t) = \frac{v(t)}{R} = \sqrt{2}\cos(\omega t)\ \text{[A]}$$

$$p(t) = \left[\sqrt{2}\cos(\omega t)\right]\cdot\left[\sqrt{2}\cos(\omega t)\right]$$

$$= \cos(2\omega t) + 1 \quad \text{[W]}$$

$$\langle p(t)\rangle = \qquad + 1 \quad \text{[W]}$$

V_{dc} = 1 [V]



I_{dc} = 1 [A]

$$V_{dc} = 1\ \text{[V]} \qquad I_{dc} = 1\ \text{[A]}$$

$$\langle p(t)\rangle = V_{dc}I_{dc} = 1\ \text{[W]}$$

1 V_{rms} → 1 Ω → 1 A_{rms} → 1 W

Same formula for all waves

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- ~~Review: Product of complex numbers~~
- ~~Product of sinusoidal waves~~
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- Exercises

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Phasors and Waves

Phasors

$\overline{V}_{rms} = 1\angle 0°$ [V]

$\overline{I}_{rms} = 1\angle 0°$ [A]

product

???

Product is a <u>complex</u> number

Cannot have unit of Watt

Equivalence

Waves

$v(t) = \sqrt{2}\cos(\omega t)$ [V]

$i(t) = \sqrt{2}\cos(\omega t)$ [A]

product

Instantaneous Power

$$p(t) = \left[\sqrt{2}\cos(\omega t)\right]\cdot\left[\sqrt{2}\cos(\omega t)\right]$$

$$= \cos(2\omega t) + 1 \quad \text{[W]}$$

average

Average Power

$$\langle p(t)\rangle = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t = 1 \text{ [W]}$$

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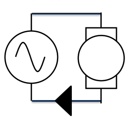
General Expression for Average Power

$\overline{V}_{rms} = V\angle\theta_v$



$\overline{I}_{rms} = I\angle\theta_i$

$$v(t) = \sqrt{2}V\cos(\omega t + \theta_v)$$

$$i(t) = \sqrt{2}I\cos(\omega t + \theta_i)$$

$$p(t) = v(t)i(t)$$

$$= 2VI\cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$= VI\left[\cos(2\omega t + \theta_v + \theta_i) \quad \text{(AC)}\right.$$

$$\left. + \cos(\theta_v - \theta_i)\right] \quad \text{(DC)}$$

- Sum & difference
- Factor of 2

$$\langle p(t)\rangle = VI\cos(\theta_v - \theta_i) = \mathrm{Re}\left\{\overline{V}_{rms}\overline{I}^*_{rms}\right\}$$

$$\overline{V}_{rms}\overline{I}^*_{rms} = (VI)e^{j(\theta_v - \theta_i)}$$

$$= VI[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)]$$

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Phasors and Waves: Complete Picture

Phasors

$\overline{V}_{rms} = V\angle\theta_v$ [V]

$\overline{I}_{rms} = I\angle\theta_i$ [A]

product

Complex Power [VA]

$$\overline{S} = \overline{V}_{rms}\overline{I}^*_{rms}$$

real part

Real Power [W]

$$P = \mathrm{Re}\left\{\overline{V}_{rms}\overline{I}^*_{rms}\right\}$$

Equivalence

Equal

Waves

$v(t) = \sqrt{2}V\cos(\omega t + \theta_v)$ [V]

$i(t) = \sqrt{2}I\cos(\omega t + \theta_i)$ [A]

product

Instantaneous Power [W]

$$p(t) = VI\left[\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right]$$

average

Average Power [W]

$$\langle p(t)\rangle = VI\cos(\theta_v - \theta_i)$$

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Phasors and Waves: Complete Picture

Phasors

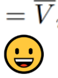
$\overline{V}_{rms} = V\angle\theta_v$ [V]

$\overline{I}_{rms} = I\angle\theta_i$ [A]

product

Complex Power [VA]

$$\overline{S} = \overline{V}_{rms}\overline{I}^*_{rms}$$



real part

Real Power [W]

$$P = \mathrm{Re}\left\{\overline{V}_{rms}\overline{I}^*_{rms}\right\}$$

Intuitive & hard to get wrong

Equivalence

Equal

Waves

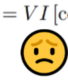
$v(t) = \sqrt{2}V\cos(\omega t + \theta_v)$ [V]

$i(t) = \sqrt{2}I\cos(\omega t + \theta_i)$ [A]

product

Instantaneous Power [W]

$$p(t) = VI\left[\cos(2\omega t + \theta_v + \theta_i) + \cos(\theta_v - \theta_i)\right]$$



average

Average Power [W]

$$\langle p(t)\rangle = VI\cos(\theta_v - \theta_i)$$

Complicated & Error-prone

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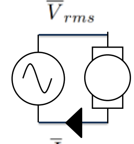
Comparing AC to DC

$\overline{V}_{rms}$

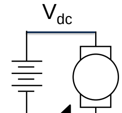


$\overline{I}_{rms}$

AC Average Power

$$P = \langle p(t)\rangle = \mathrm{Re}\left\{\overline{V}_{rms}\overline{I}^*_{rms}\right\}$$

V_{dc}



I_{dc}

DC Power

$$P = \langle p(t)\rangle = V_{dc}I_{dc}$$

DC formula is a <u>special case</u> of the AC formula

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Power Triangle

Power engineer's convention.
Make sure you memorize this slide

$$\overline{S} = P + jQ = S\measuredangle\phi$$

Complex Power [VA]

$$\overline{S} = \overline{V}_{rms}\overline{I}^*_{rms}$$

Reactive Power [VAR]

$$Q = \text{Im}\{\overline{V}_{rms}\overline{I}^*_{rms}\}$$

Apparent Power [VA]

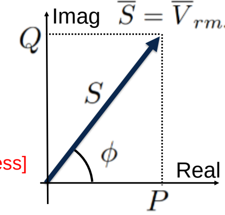
$$S = |\overline{S}|$$

Power Factor [Dimensionless]

$$\cos\phi = P/S$$

$\phi > 0$ Lagging

$\phi < 0$ Leading



Real Power [W]

$$P = \text{Re}\{\overline{V}_{rms}\overline{I}^*_{rms}\}$$

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Today

- ~~Review: Product of complex numbers~~
- ~~Product of sinusoidal waves~~
- ~~Complex power and the power triangle~~
- Exercises

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Exercise 1

$\overline{V} = 10\measuredangle 15^\circ$ [V] $\quad \overline{I} = 5\measuredangle -30^\circ$ [A]

What is the REAL power?

A) 50 W

B) 35.355 W

C) 48.30 W

D) 70.71 W

Hint: Phasors are <u>always</u> in RMS

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Exercise 1

$\overline{V} = 10\measuredangle 15^\circ$ [V] $\quad \overline{I} = 5\measuredangle -30^\circ$ [A]

What is the REAL power?

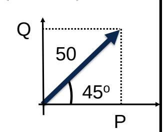
~~A) 50 W~~

B) 35.355 W

~~C) 48.30 W~~

~~D) 70.71 W~~

$$\overline{S} = \overline{V}\overline{I}^* \quad \text{by definition}$$
$$= (10\measuredangle 15^\circ)(5\measuredangle -30^\circ)^*$$
$$= (10\measuredangle 15^\circ)(5\measuredangle 30^\circ)$$
$$= 50\measuredangle 45^\circ$$



Hint: Phasors are <u>always</u> in RMS

$$P = \text{Re}\{\overline{S}\} = 50\cos(45^\circ)$$

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Exercise 2

$\overline{V} = 10\measuredangle 15^\circ$ [V] $\quad \overline{I} = 5\measuredangle -30^\circ$ [A]

What is the REACTIVE power?

A) 50 W

B) 35.355 W

C) -12.94 VAR

D) 35.355 VAR

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Exercise 2

$\overline{V} = 10\measuredangle 15^\circ$ [V] $\quad \overline{I} = 5\measuredangle -30^\circ$ [A]

What is the REACTIVE power?

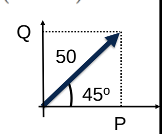
~~A) 50 W~~

~~B) 35.355 W~~

~~C) -12.94 VAR~~

D) 35.355 VAR

$$\overline{S} = \overline{V}\overline{I}^* \quad \text{by definition}$$
$$= (10\measuredangle 15^\circ)(5\measuredangle -30^\circ)^*$$
$$= (10\measuredangle 15^\circ)(5\measuredangle 30^\circ)$$
$$= 50\measuredangle 45^\circ$$



No imaginary watts!
Watch your units

$$Q = \text{Im}\{\overline{S}\} = 50\sin(45^\circ)$$

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Exercise 3

$$v(t) = \sqrt{2}(10)\cos(\omega t + 15^\circ)$$
$$i(t) = \sqrt{2}(7)\sin(\omega t + 75^\circ)$$

What is the COMPLEX power?

A) 70+j0 VA
B) 35-j60.62 VA
C) 60.62+j35 VA
D) 121.24+j70 VA

Hint:
$$\sin\theta = \cos(\theta - 90^\circ)$$

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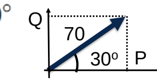
Exercise 3

$$v(t) = \sqrt{2}(10)\cos(\omega t + 15^\circ) \quad \overline{V} = 10\angle 15^\circ$$
$$i(t) = \sqrt{2}(7)\sin(\omega t + 75^\circ) \quad \overline{I} = 7\angle(75^\circ - 90^\circ)$$
$$= 7\angle(-15^\circ)$$

What is the COMPLEX power?

~~A) 70+j0 VA~~
~~B) 35-j60.62 VA~~
C) 60.62+j35 VA
~~D) 121.24+j70 VA~~

$$\overline{S} = \overline{V}\overline{I}^*$$
$$= (10\angle 15^\circ)(7\angle -15^\circ)^*$$
$$= (10\angle 15^\circ)(7\angle 15^\circ)$$
$$= 70\angle 30^\circ$$



Hint:
$$\sin\theta = \cos(\theta - 90^\circ)$$

$$P = \text{Re}\{\overline{S}\} = 70\cos(30^\circ)$$
$$Q = \text{Im}\{\overline{S}\} = 70\sin(30^\circ)$$

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