ECE 330 (Spring 2016) **Instructors:** A. Flifet and S. Bose

Midterm 2

Duration: 90 minutes Total points: 100

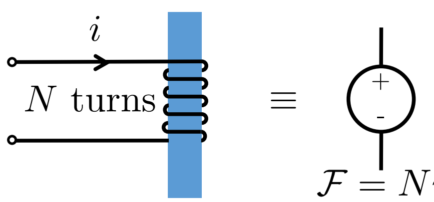
Name: ____.

Section (Tick one): C (Mon/Wed/Fri) _______ F (Tue/Thu) _______.

Scores (For official use only):

Problem 1:_________/25, Problem 2:_________/25,
Problem 3:_________/25, Problem 4:_________/25. **Total score:**_____________/100.

Relevant formulae



$\mathcal{F} = Ni$ $\qquad \lambda = Li(\text{if linear})$ $\qquad v = \frac{d\lambda}{dt}$

$W_m(\lambda, x) = \int_0^\lambda i(\hat{\lambda}, x) d\hat{\lambda}$ $\quad W'_m(i, x) = \int_0^i \lambda(\hat{i}, x) d\hat{i}$ $\quad \lambda = \frac{\partial W'_m(i,x)}{\partial i}$ $\quad i = \frac{\partial W_m(\lambda,x)}{\partial \lambda}$

$f^e(\lambda, x) = -\frac{\partial W_m(\lambda,x)}{\partial x}$ $\quad f^e(i, x) = \frac{\partial W'_m(i,x)}{\partial x}$ $\quad T^e(\lambda, \theta) = -\frac{\partial W_m(\lambda,\theta)}{\partial \theta}$ $\quad T^e(i, \theta) = \frac{\partial W'_m(i,\theta)}{\partial \theta}$

$W_m + W'_m = \lambda i$ $\quad EFE_{a\to b} = \int_a^b i d\lambda$ $\quad EFM_{a\to b} = -\int_a^b f^e dx$ $\quad EFM_{a\to b} = -\int_a^b T^e d\theta$

$$EFE_{a\to b} + EFM_{a\to b} = W_m|_b - W_m|_a$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \to \underline{x}(t + \Delta t) \approx \underline{x}(t) + \Delta t \underline{f}(\underline{x}(t), \underline{u}(t)) \text{ and } \underline{0} = \underline{f}(\underline{x}_e, \underline{u}_e)$$

$$\begin{aligned} &X = (x_1, \ldots, x_n)^\top, \\ &\dot{x}_i = f_i(x_1, \ldots, x_n) \text{ for } i = 1, \ldots, n \end{aligned} \implies \Delta\dot{x}_i = \sum_{k=1}^n \left.\frac{\partial f_i}{\partial x_k}\right|_{X^e} \Delta x_i \text{ for } i = 1, \ldots, n.$$

$$\begin{aligned} &\dot{X} = AX, \\ &\text{eigs}(A) = \{\lambda_1, \ldots, \lambda_n\} \end{aligned} \implies \begin{cases} \text{Re}\{\lambda_i\} < 0 \;\forall\, i \implies \text{stable} \\ \text{Re}\{\lambda_i\} > 0 \text{ for any } i \implies \text{unstable} \\ \text{Re}\{\lambda_i\} \le 0 \;\forall\, i, \text{ and Re}\{\lambda_i\} = 0 \text{ for some } i \implies \text{marginally stable.} \end{cases}$$

Problem 1 [25 points]

The flux linkages in an energy-conservative electromechanical system are given by

$$\begin{aligned}
\lambda_a &= L_a i_a + (M\cos\theta)i_b + (M\sin\theta)i_c,\\
\lambda_b &= L_b i_b + (M\cos\theta)i_a,\\
\lambda_c &= L_c i_c + (M\sin\theta)i_a,
\end{aligned}$$

where L_a, L_b, L_c and M are positive constants, and i_a, i_b, i_c are currents into the system.

(a) Is the system electrically linear? [1 point]

(b) How many electrical and mechanical ports does the system have? [2 + 2 points]

(c) Find the co-energy $W'_m(i_a, i_b, i_c, \theta)$ for this system. [12 points]

(d) Compute the torque of electric origin $T_e(i_a, i_b, i_c, \theta)$. [3 points]

(e) Compute the maximum absolute value of $T_e(i_a = I, i_b = I, i_c = I, \theta)$ over $\theta \in [0, 2\pi]$, where I is a positive constant. Also, report *all* values of $\theta \in [0, 2\pi]$, where this maximum is attained. [5 points]

Problem 2 [25 points]

The flux linkage in an energy-conservative electromechanical system is given by

$$\lambda(i, x) = \frac{\gamma i}{x - x_0},$$

where i denotes the current into the system and x defines the geometry of the mechanical subsystem. Also, $x_0 = 1$ m, and $\gamma = 4$ Wb-turns-m. The system is being operated on the closed cycle a - b - c - d - a as indicated in the figure below.

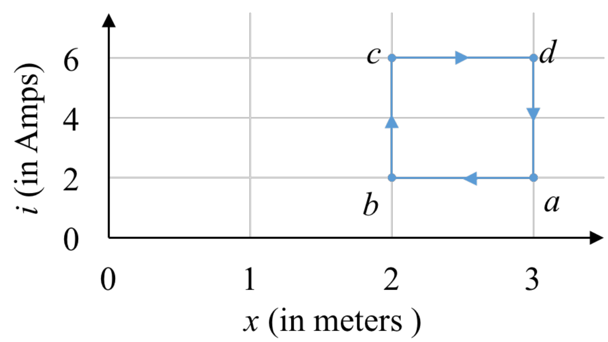


Figure 1: Cycle over which the electromechanical system is operated

(a) Compute the co-energy $W'_m(i, x)$ in terms of γ and x_0. [3 points]

(b) Compute the force of electrical origin $f_e(i, x)$ in terms of γ and x_0. [3 points]

(c) Compute the numerical values of the energy stored in the coupling field W_m at points a and c as indicated in the above figure. [6 points]

(d) Compute the "energy from mechanical" over the cycle shown in the figure, i.e., $\text{EFM}|_{\text{cycle}}$. [12 points]

(e) Based on your answer in part (d), state whether the electromechanical system is operating as a motor or a generator over the cycle. [1 points]

Problem 3 [25 points]

A dynamical system is described by the following differential equation:

$$\ddot{x} + \dot{x} + x = x^2.$$

(a) What is the state space model for the above dynamical system? [3 points]

(b) Find all the equilibrium points of the system in its state space form. [4 points]

(c) Linearize the state space model (you derived in part (a)) around each equilibrium point (you derived in part (b)). Your linearized systems should be given in state space form. [8 points]

(d) Find whether the linearized dynamical systems (that you derived in part (c)) are stable, unstable, or marginally stable at the origin. [10 points]

Problem 4 [25 points]

The state space model of a dynamical system is described by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = 3 - x_1^2.$$

(a) What is the order of the dynamical system in the above state space representation? [1 point]

(b) Is the dynamical system linear? [1 point]

(c) Find all its equilibrium points. [4 points]

(d) Given the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$ at $t = 0$, fill in the missing entries in Table 1 using Euler's method for numerical integration with a time step of $\Delta t = 0.1$. [12 points]

t	$x_1^{\text{Euler}}(t)$	$x_2^{\text{Euler}}(t)$
0	1	0
0.1		
0.2		
0.3		

Table 1: Euler's method with $\Delta t = 0.1$

(e) For the above dynamical system with the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$ at $t = 0$, the exact solution for $x_1(t)$ is given in Table 2 for $t = 0, 0.1, 0.2$, and 0.3. Compute the absolute value of the percentage error of $x_1(t)$ from Euler's method (you derived in part (d)) against the exact solution to populate the missing entries in Table 2.

t	$x_1^{\text{exact}}(t)$	$x_1^{\text{Euler}}(t)$	$\frac{\lvert x_1^{\text{Euler}}(t) - x_1^{\text{exact}}(t)\rvert}{x_1^{\text{exact}}(t)} \times 100\%$
0	1	1	0
0.1	1.01		
0.2	1.04		
0.3	1.09		

Table 2: Comparing Euler's method to the exact solution

Based on your calculations, state whether the percentage error of the solution from Euler's method is increasing or decreasing with t? [4 + 1 points]

(f) Recall that you used $\Delta t = 0.1$ to calculate $x_1^{\text{Euler}}(t)$. Do you expect the absolute value of the percentage error in Table 2 to increase or decrease if we change Δt to 0.01, instead of 0.1? State your reason. [1 + 1 points]

Problem 1.

a. Yes, since the flux linkages $\lambda = \begin{pmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{pmatrix}$ varies linearly with the currents $i = \begin{pmatrix} i_a \\ i_b \\ i_c \end{pmatrix}$.

b. # electrical ports = 3
mechanical ports = 1.

c. $W_m'(i_a, i_b, i_c, \theta)$

$$= \int_0^{i_a} \lambda_a(i_a', 0, 0, \theta)\, di_a' + \int_0^{i_b} \lambda_b(i_a, i_b', 0, \theta)\, di_b' + \int_0^{i_c} \lambda_c(i_a, i_b, i_c', \theta)\, di_c'$$

$$= \int_0^{i_a} L_a i_a'\, di_a' + \int_0^{i_b} \left(L_b i_b' + (M\cos\theta)\, i_a \right) di_b' + \int_0^{i_c} \left(L_c i_c' + (M\sin\theta)\, i_a \right) di_c'$$

$$= \frac{1}{2} L_a i_a^2 + \frac{1}{2} L_b i_b^2 + M\cos\theta\, i_a i_b + \frac{1}{2} L_c i_c^2 + M\sin\theta\, i_a i_c.$$

d. $T_e(i_a, i_b, i_c, \theta) = \dfrac{\partial W_m'}{\partial \theta}$

$$= -M\sin\theta\, i_a i_b + M\cos\theta\, i_a i_c.$$

e. $T_e(I, I, I, \theta) = MI^2(\cos\theta - \sin\theta) := f(\theta)$

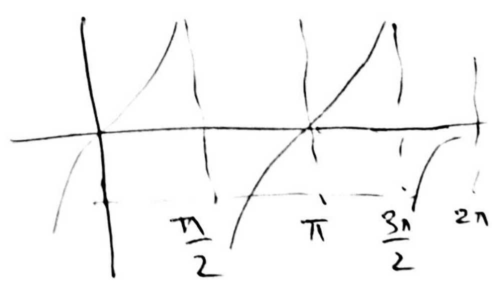
$$\frac{\partial f}{\partial \theta} = MI^2(-\sin\theta - \cos\theta)$$

$$\frac{\partial^2 f}{\partial \theta^2} = MI^2(-\cos\theta + \sin\theta)$$

Candidate solutions for $\max|f(\theta)|$ are 0, 2π, and θ^* such that $\left.\dfrac{\partial f}{\partial \theta}\right|_{\theta^*} = 0$

<u>Solving for θ^*:</u>

$$\left.\frac{\partial f}{\partial \theta}\right|_{\theta^*} = -\sin\theta^* - \cos\theta^* = 0 \Rightarrow \tan\theta^* = -1$$



$$\Rightarrow \theta^* = \pi - \frac{\pi}{4},\ 2\pi - \frac{\pi}{4}.$$

$$= \frac{3\pi}{4},\ \frac{7\pi}{4}.$$

$|f(0)| = MI^2$

$|f(2\pi)| = MI^2$

$\left|f\left(\frac{3\pi}{4}\right)\right| = \left|MI^2\left(\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right)\right| = \sqrt{2}\,MI^2$

$\left|f\left(\frac{7\pi}{4}\right)\right| = \left|MI^2\left(\cos\frac{7\pi}{4} - \sin\frac{3\pi}{4}\right)\right| = \sqrt{2}\,MI^2$

$\Rightarrow \max_{\theta\in[0,2\pi]} T^e(I,I,I,\theta) = \sqrt{2}\,MI^2$ occurs at $3\pi/4$, $7\pi/4$.

Problem 2.

a. $W_m'(i, x) = \int_0^i \lambda(i', x)\, di'$

$$= \int_0^i \frac{\gamma i'}{x - x_0}\, di' = \frac{\gamma}{x - x_0} \cdot \frac{i^2}{2}.$$

b. $f_e(i, x) = \frac{\partial W_m'}{\partial x} = \frac{\gamma i^2}{2}\left[\frac{-1}{(x - x_0)^2}\right] = \frac{-\gamma i^2}{2(x - x_0)^2}.$

c. The numerical value of W_m and W_m' are the same for an electrically linear system.

$\therefore$ W_m at point c = W_m' at pt. c

$$= \frac{4^2}{\cancel{2}-1} \cdot \frac{6^2}{2} \text{ J}$$

$$= 72 \text{ J}$$

W_m at point a $= \frac{4}{3-1} \cdot \frac{2^{\cancel{2}}}{\cancel{2}}$ J

$$= 4 \text{ J}.$$

d. $$EFM\Big|_{cycle} = \int_{cycle} -f^e\,dx$$

$$= \int_a^b -f^e(i,x)\,dx + \int_c^d -f^e(i,x)\,dx$$

$\left[\because x \text{ does not change over } \begin{matrix} b\to c \\ \text{and } d\to a \end{matrix}\right].$

$$= \int_{x=3}^{x=2} \frac{\overset{=4}{\cancel{\gamma}}\cdot 2^2}{2(x-x_0)^2}\,dx + \int_{x=2}^{x=3} \frac{\overset{=4}{\gamma}\cdot 6^2}{2(x-x_0)^2}\,dx$$

$$= 8\left(\frac{-1}{x-\underset{=1}{\cancel{x_0}}}\right)\Bigg|_{x=3}^{x=2} + 72\cdot\left(\frac{-1}{x-\underset{=1}{\cancel{x_0}}}\right)\Bigg|_{x=2}^{x=3} \quad J$$

$$= 8\left[\frac{1}{3-1} - \frac{1}{2-1}\right] + 72\left[\frac{1}{2-1} - \frac{1}{3-1}\right]. \quad J$$

$$= 8\left(\frac{-1}{2}\right) + 72\left(\frac{1}{2}\right) \quad J$$

$$= 32 \text{ J}.$$

e. $EFM\Big|_{cycle} > 0 \Rightarrow$ Generator.

Problem 3.

$$\ddot{x} + \dot{x} + x = +x^2.$$

a. Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 = x$, $x_2 = \dot{x}$.

$$\therefore \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - x_1 + x_1^2 \end{pmatrix}.$$

b. Let $X^e = \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix}$ be an eq. pt.

$$\Rightarrow x_2^e = 0 \quad \text{and} \quad -\underbrace{x_2^e}_{=0} - x_1^e + (x_1^e)^2 = 0.$$

$$\Rightarrow x_1^e = 0, 1.$$

$\therefore$ eq. pts are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

c. Linearizing around $X^e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let $Y = X - X^e$.

$$\Rightarrow \dot{Y} = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{X^e} & \left.\frac{\partial f_1}{\partial x_1}\right|_{X^e} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{X^e} & \left.\frac{\partial f_2}{\partial x_2}\right|_{X^e} \end{pmatrix} \cdot Y, \quad \text{where } \begin{array}{l} f_1(x_1, x_2) = x_2 \\ f_2(x_1, x_2) = -x_2 - x_1 + x_1^2. \end{array}$$

$2x_1 - 1$

$$= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \cdot Y.$$

Linearizing around $x^e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} y.$$

d. Eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ are roots of

$$\det(A - \lambda I) = 0$$

iff $\det \begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = 0$.

iff $(-\lambda)(-\lambda-1) - 1(-1) = 0$

iff $\lambda^2 + \lambda + 1 = 0$.

iff $\lambda_i = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1}{2} \pm j\dfrac{\sqrt{3}}{2}$.

$Re\{\lambda_i\} = \dfrac{-1}{2}$ for $i = 1, 2$.

< 0.

$\Rightarrow$ Stable.

Eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ are roots of

$\det(A - \lambda I) = 0$.

iff $\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda-1 \end{pmatrix} = 0$

iff $(-\lambda)(-\lambda-1) - 1 = 0$

iff $\lambda^2 + \lambda - 1 = 0$

$\Rightarrow \lambda = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

$\lambda = \dfrac{\sqrt{5}-1}{2} > 0$

$\Rightarrow$ Unstable.

Problem 4.

<u>a.</u> Order of dynamical system = 2.

<u>b.</u> No., since $\dot{x}_2 = 3 - x_1^2$ is not linear in x_1.

<u>c.</u> Let $X^e = \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix}$ be an eq. pt.

$$\Rightarrow \quad x_2^e = 0 \quad \text{and} \quad 3 - (x_1^e)^2 = 0.$$

$$\Rightarrow \quad x_1^e = \pm\sqrt{3}.$$

$\therefore$ eq. pts are $\begin{pmatrix} \pm\sqrt{3} \\ 0 \end{pmatrix}$.

<u>d.</u> $x_1(t_{k+1}) = x_1(t_k) + \Delta t \cdot x_2(t_k)$

and $x_2(t_{k+1}) = x_2(t_k) + \Delta t \cdot \left[3 - x_1^2(t_k)\right]$.

t	$x_1^{\text{Euler}}(t)$	$x_2^{\text{Euler}}(t)$
0	1	0.
0.1	1	0.2
0.2	1.02	0.4
0.3	1.06	0.6.

e.

t	$x_1^{exact}(t)$	$x_1^{Euler}(t)$	% error
0	1	1	0
0.1	1.01	1	0.99
0.2	1.04	1.02	1.92
0.3	1.09	1.06	2.75.

The percentage error is increasing with time.

~~In general, error percentage should grow with time, because Euler's method is an approximate method, whose errors accumulate with iterations.~~

f. As Δt becomes smaller, Euler's method should become more and more accurate. This is so, because the derivative $\dot{x}_i(t_k)$ is better approximated by $\frac{x_i(t_{k+1}) - x_i(t_k)}{t_{k+1} - t_k}$ as $\Delta t = t_{k+1} - t_k$ becomes smaller and smaller.